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ABSTRACT

This is part two of a two-part SMSG textbook for inservice education of elementary teachers. Part two contains chapters 6 through 14 of Unit B: Addition and Multiplication; Subtraction and Division; Numeration: Naming Numbers; Premeasurement Concepts; Addition and Subtraction Techniques; Introducing Rational Numbers; Measurement; Multiplication and Division Techniques; and Structure. (MK)

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INSERVICE COURSE FOR ELEMENTARY SCHOOL TEACHERS

SPECIAL EDITION (Preliminary)

Part 2

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2

INSERVICE COURSE FOR ELEMENTARY SCHOOL TEACHERS

SPECIAL EDITION (Preliminary)

Part 2

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Chapter 6

ADDITION AND MULTIPLICATION

Number Property of a Union

In Chapter 4, was introduced the concept of an operation. An operation assigns to each member of a set a unique element of a second set. Here, it was pointed out, for example, that the number property of a set is an operation. In this case, a member of the first set is a set, and a member of the second set is a number. Essentially then if P is a set whose members are sets, and W is the set of whole numbers, the mechanics in this operation may be illustrated as follows:

P = {{•, ☽, □}, {a, b}, {Lorie, Peggy, Rosie}, {c, l, i, n, e}, ...}

W = {0, 1, 2, 3, 4, 5 ...}

Operations were defined on two sets of sets. Then, we were considering two sets, say, S and T as we have below. Each member of S is a set and each member of T is a set. A pair of sets, one from S and one from T, is assigned a unique element. The unique element may be a set whose elements are members of the initial sets as in the case of the union or intersection; the unique element may be a set whose elements are compositions as in the case of the product set. This is what we mean when we say that a third set is created from two given sets.

If A is one of the sets belonging to, say, S, and B is one of the sets belonging to T, the set operation may assign a unique element C to the combination of A and B. To illustrate, suppose S and T are as follows:

S = {{a, b, c}, {α, β}, {Lorie, Peggy, Rosie}, {α, γ, f, elephant}, ...}

T = {{Harry, Karla, Pat, Charles}, {e, g, b}, {m, e, r, v}, ...}

If A = {Lorie, Peggy, Rosie} and B = {m, e, r, v}, then A is one of the sets of S and B is one of the sets of T. By the operation of union, a set

$$C = \{\text{Lorie, Peggy, Rosie}, m, e, r, v\}$$

is assigned to the combination denoted as $A \cup B$.

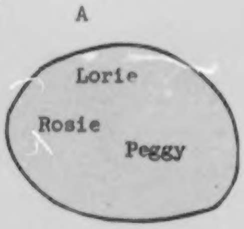
Our immediate purpose is to tie such set operations to operations with numbers. Let us examine first, the number properties of sets and their union. To illustrate, the union and the number properties may be:

$$\underset{\substack{\downarrow\\3}}{\{\text{Lorie, Peggy, Rosie}\}} \cup \underset{\substack{\downarrow\\4}}{\{m, e, r, v\}} = \underset{\substack{\downarrow\\7}}{\{\text{Lorie, Rosie, Peggy}, m, e, r, v\}}$$

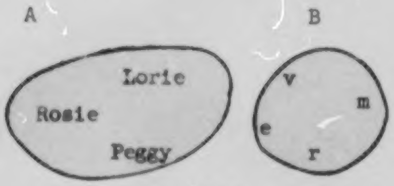
Consider a second example:

$$\underset{\substack{\downarrow\\2}}{\{\alpha, \beta\}} \cup \underset{\substack{\downarrow\\4}}{\{\alpha, \gamma, f, \text{elephant}\}} = \underset{\substack{\downarrow\\5}}{\{\alpha, \beta, \gamma, f, \text{elephant}\}}$$

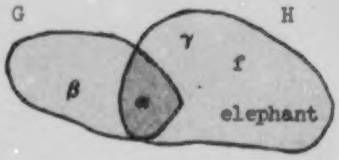
From the number properties indicated in the first instance, we may recognize the familiar 3, 4, 7 combination as one of the combinations in addition whereas, the combination 2, 4, 5 is not so recognized. It should be clear that this is because we have a union of disjoint sets in the former and that the sets are not disjoint in the latter. To visualize this, we may enclose all the members of a particular set within some boundary, as for example,



Then, the two unions may be illustrated as follows:



$A \cup B$



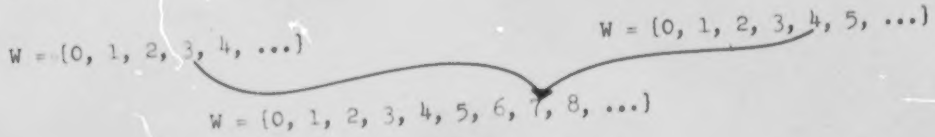
$G \cup H$

Observe the overlapping of G and H. The element common to both G and H is α. The point we make at this time is that the ordinary arithmetic operation of addition is deduced from the union of two disjoint sets. If N(A) is the number property of A and N(B) is the number property of B, then the number property of the union gives the result of adding the two numbers, N(A) and N(B), provided A and B are disjoint. In other words,

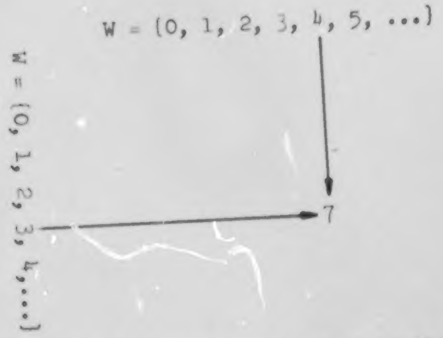
if A and B are disjoint sets, then $N(A) + N(B) = N(A \cup B)$.

The union is an operation on sets and addition is an operation on numbers. Corresponding to the union of two disjoint sets is the addition of their number properties; the sum of their number properties is the number property of the union.

Thus, by looking at the unions of disjoint sets, the addition operation is defined on whole numbers


W = {0, 1, 2, 3, 4, ...}
W = {0, 1, 2, 3, 4, 5, ...}
W = {0, 1, 2, 3, 4, 5, 6, 7, 8, ...}

The diagram that we have above, indicating how 7 is produced as the unique result of combining 3 and 4 in this operation may be rearranged slightly as below:



Clearly, this is the arrangment indicated by the usual basic addition table in which the sum of 3 and 4 is located as the entry on the fourth row and fifth column.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

$3 + 4 = 7$

Problems[*]

1. If A and B are as below, find $A \cup B$.
   a. $A = \{1, 2\}$; $B ; \{3\}$
   b. $A = \{1, 2, 3\}$; $B = \{2, 4, 6\}$
   c. $A = \{1, 2, 3\}$; $B = \{\ \}$
   d. $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 3, 5, 7, 9\}$
   e. $A = \{1, 3, 5, 7, 9\}$; $B = \{1, 2, 3, 4, 5\}$
   f. $A = \{a, b, c, d\}$; $B = \{\alpha, \beta, \gamma, \delta, \epsilon\}$

2. For each of the sets in Problem 1, find $N(A)$, $N(B)$, and $N(A \cup B)$. State whether it is true that $N(A) + N(B) = N(A \cup B)$ for each pair of sets; explain why this equality holds or why it doesn't hold.

3. Draw diagrams to represent the union of the following sets.
   a. $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 3, 5, 7, 9\}$
   b. $A = \{1, 2, 3, 4\}$; $B = \{5, 6, 7\}$
   c. A = {calf, camel, caribou, cougar, cow, coyote}; B = {bull, calf, steer, cow, ox}

4. Suppose R is the set of numbers listed in the row headings of the addition table; R may be described as the set of possible addends. If C is the set of numbers listed in the column headings and B is the set of numbers in the main body of the table for addition, how would you describe C and B?

---

[*] Solutions for problems in this chapter are on page 182.

Properties under Addition

Since addition arises from the union of sets, we can expect that properties under the union operation may have implications for the addition operation. We observe first, that the union of two sets is a set. This, of course, is from the definition of union. As a whole number may be assigned to any set, corresponding to the fact that

the union of two sets is a set,

we have

the sum of two whole numbers is a whole number.

Both of these are statements of closure properties. The first is the closure property of sets under union, and the second is the *closure property of whole numbers under addition*. If an operation that is defined on a set is such that the result is an element of the same set, then we say that the set is closed under the operation. For example, if we consider the operation described by "double the number", then the result of doubling any whole number is also a whole number. We visualize this operation thus:

$$\begin{tikzcd}[column sep=tiny] W = \{0, & 1, & 2, & 3, \arrow[rrr, bend left] & 4, & 5, & 6, & 7, & 8, & 9, & \ldots\}, \end{tikzcd}$$

showing, for instance, that if 3 is a member of W, doubling 3 is also a member of W.

Similarly, we may visualize closure under addition thus:

$$\begin{tikzcd}[column sep=tiny] W = \{0, & 1, & 2, & 3, \arrow[rrrrr, bend left] & 4, & 5, \arrow[rrr, bend left] & 6, & 7, & 8, & 9, & \ldots\}, \end{tikzcd}$$

showing here, that the result of 3 and 5 is an element of W. Roughly, this means that we don't have to reach outside the set for the result under the operation. A consequence of this property is that we may repeat the operation on the sum.

Another property under the union pertains to the order of operation. If A and B are sets, the result of joining A to B is the same as joining B to A. We summarize this by saying that the union is a commutative operation. For any sets A and B,

$$A \cup B = B \cup A.$$

Corresponding to this, we have the *commutative property of whole numbers under addition*. For any whole numbers *a* and b,

$$a + b = b + a.$$

For instance, the sum of 3 and 4 (which may be written $3 + 4$) and the sum of 4 and 3 (written $4 + 3$) both yield the same number, 7. For this reason, we can write

$$3 + 4 = 4 + 3 .$$

Both $3 + 4$ and $4 + 3$ name the same number.

We have said above that a consequence of the closure property under addition is that the operation may be repeated on the sum. For example, since $3 + 4$ is a whole number we might add another whole number say, 9, to the sum. This would be indicated in the grouping of $3 + 4$ in parentheses, thus:

$$(3 + 4) + 9 .$$

Since the sum of $3 + 4$ is 7, the expression $(3 + 4) + 9$ means the sum of $7 + 9$; or, in other words, 16. That is to say,

$$(3 + 4) + 9 = 7 + 9 \text{ and } 7 + 9 = 16;$$

therefore, $(3 + 4) + 9 = 7 + 9$
$= 16.$

Since 16 is a whole number, this process may be continued as needed. Thus, we may add say, 5, to the result of $(3 + 4) + 9$ to get the result of $((3 + 4) + 9) + 5$, which is the same as $16 + 5$, or 21.

Our next concern is to pursue the concept of grouping the addends. Recall that for sets, the grouping under the union did not change the resulting set. That is, the union is said to be an associative operation. Consequently, both $(A\cup B)\cup C$ and $A\cup(B\cup C)$ give rise to the same number property. Therefore, we have the *associative property of whole numbers under addition*:

for whole numbers *a*, b, and c,
$$(a + b) + c = a + (b + c).$$

If A has the number property 3, B has the number property 4, and C has the number property 9, then $A\cup B$ has the number property 7 and $(A\cup B)\cup C$ has the number property $7 + 9$, or 16. For these same sets, $B\cup C$ has the number property 13 and $A\cup(B\cup C)$ has the number property $3 + 13$, or 16. A, B, C are of course, all disjoint since addition is derived from the union of disjoint sets. To trace "the machinery" behind this property, we can display $(a + b) + c$ and $a + (b + c)$ as follows:

$$\begin{array}{ccc} (3 + 4) + 9 & & 3 + (4 + 9) \\ \| & & \| \\ 7 + 9 & & 3 + 13 \\ \| & & \| \\ 16 & = & 16 \end{array}$$

with the vertical slashes indicating equality as we read vertically This may be interpreted as follows:

$$(3 + 4) + 9 = 7 + 9 = 16;$$

independently $\quad 3 + (4 + 9) = 3 + 13 = 16$.

Since $16 = 16$, we can follow the chain thus:

$$(3 + 4) + 9 \rightarrow 7 + 9 \rightarrow 16 \rightarrow 16 \rightarrow 3 + 13 \rightarrow 3 + (4 + 9).$$

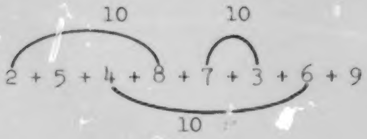
From this, we conclude that $(3 + 4) + 9 = 3 + (4 + 9)$. The associative property states that this characteristic is not restricted to just the numbers 3, 4 and 9; it holds for any whole numbers a, b, and c; that is, $(a + b) + c = a + (b + c)$.

The property for closure allows us to repeatedly add as many numbers as we wish. The commutative and associative properties allow us to do the adding in whichever way we please, as long as each addend is appropriately accounted for. For example, we may require the sum:

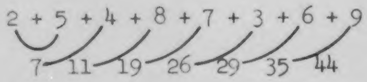
$$2 + 5 + 4 + 8 + 7 + 3 + 6 + 9.$$

Closure states that this can be done; merely add any two, then continue to add any of the other addends to the result and so on. Commutativity and associativity say that if we so choose, we are free to pick appropriate combinations at will.

For instance, in the above example, it may be desirable to look for combinations of ten since adding one ten to another is easy for us. For the above sum, we may then find it convenient to group in the following way: $(2 + 8)$, $(4 + 6)$, $(7 + 3)$. Hence, the scheme of our procedure is:



From this, all we need is the sum of 5 and 9, then add 30 to this sum, getting 44 as the final result. However, we may proceed directly from one addend to another:



Any appropriate way we choose should yield the same sum. In the name of efficiency if not of sanity, the first method is more likely to be preferred.

Let us examine how we make use of the commutative and associative properties. We shall not trace through every step involved; rather, we shall indicate some of the bigger steps typical of the situation. Suppose we want the sum of 2 and 8. Because of commutativity, we may first interchange the order of the 4 and the 8:

$$2 + 5 + 4 + 8 + 7 + 3 + 6 + 9$$
$$= 2 + 5 + 8 + 4 + 7 + 3 + 6 + 9.$$

This may be followed next, by interchanging the order of the 5 and the 8 to get

$$2 + 8 + 5 + 4 + 7 + 3 + 6 + 9 .$$

Similarly, we can go leap-frogging for the sums of other pairs of numbers that we may choose.

So far, associativity has not been used, or so it seems. The fact is, we just conveniently neglected to mention it when it did occur. To make it easier to follow let us consider first, just the partial sum

$$2 + 5 + 4 + 8 .$$

If $2 + 5$ is obtained first, and 4 added to this result, followed by adding 8 to the result of $(2 + 5) + 4$, this may be indicated

$$((2 + 5) + 4) + 8$$

where the inner parentheses show the first grouping of 2 and 5. Thus,

$$((2 + 5) + 4) + 8 \text{ means } (7 + 4) + 8 .$$

By the associative property,

$$(7 + 4) + 8 = 7 + (4 + 8) = (2 + 5) + (4 + 8) .$$

In summary, what we are saying is

$$((2 + 5) + 4) + 8 = (2 + 5) + (4 + 8).$$

Clearly, this process may be repeated again and again. So, while the associative property had not been in evidence before, it is still very much a part of the process. This is why we say that both the commutative and associative properties are involved in our "pick and choose" process. Further analysis of the role of the associative property involves further "nesting" of parentheses, for example,

$$2 + 5 + 4 + 8 + 7 = (((2 + 5) + 4) + 8) + 7,$$

and so forth.

From the standpoint that an object (set) is produced from two sets in forming the union, we can regard the union as a binary operation; it operates on two objects to give a third. We also have noted that with closure, we may continue such an operation on the union. Moreover, because of associativity, the compound result is unique (one and only one set is defined as the union regardless of grouping). Thus, $A \cup B \cup C$ can be written without parentheses. This concept is carried over to the operation of addition, and the notation for the sum is freed of any parentheses.

Problems

5. Which of the following statements are examples of the commutative property under addition?
   - a. $7 + 8 = 8 + 7$
   - b. $7 + 8 = 7 + 8$
   - c. $(7 + 8) + 9 = (8 + 7) + 9$
   - d. $(7 + 8) + 9 = 7 + (8 + 9)$
   - e. $78 = 87$
   - f. $(7 + 8) + 9 = 9 + (7 + 8)$
   - g. $7 + 8 + 9 = 9 + 8 + 7$

6. Which of the following statements are examples of the associative property under addition?

   a. $(7 + 8) + 9 = (7 + 8) + 9$
   b. $(7 + 8) + 9 = 7 + (8 + 9)$
   c. $(7 + 8) + 9 = 9 + (7 + 8)$
   d. $7 + 8 + 9 = (7 + 8) + 9$
   e. $7 + 8 + 9 + 10 = (7 + 8) + (9 + 10)$
   f. $(7 + (8 + 9)) + 10 = ((8 + 9) + 7) + 10$
   g. $(7 + (8 + 9)) + 10 = 7 + ((8 + 9) + 10)$

7. Which property or properties of whole numbers under addition make(s) each of the following true?

   a. $(7 + 8) + (9 + 10) = (9 + 10) + (7 + 8)$
   b. $(7 + 8) + (9 + 10) = (7 + 8) + (10 + 9)$
   c. $7 + 8 = 15$
   d. $7 + 8 + 9 + 10 = 10 + 9 + 8 + 7$
   e. $7\ 8\ 9 = 9\ 8\ 7$
   f. $7 + (8 + 9) + 10 = (7 + 8) + (9 + 10)$
   g. $7 + 8 + 9 + 10 = (7 + 10) + (8 + 9)$

Another property of sets under the union operation that is significant for the addition operation is one that is connected with the union of a set with the empty set. We have observed before that if A is a set then $A \cup \{\ \} = A$. Since the number property of the empty set is 0, if the number property of A is $\underline{a}$, then the corresponding statement for the above observation is:

for any whole number $\underline{a}$,

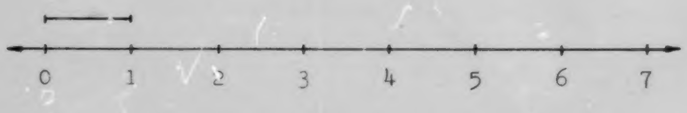
$$a + 0 = a.$$

Of course, because of the commutative property, we also have $0 + a = a$.

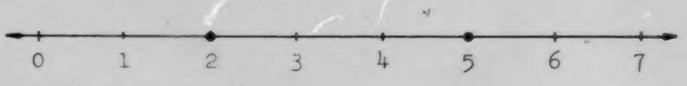
Since addition of 0 to any number produces that identical number, 0 is called the identity element with respect to addition. No other element plays this same role. The property referred to above is known as the property of zero under addition, or in short, the addition property of zero.

Addition on the Number Line

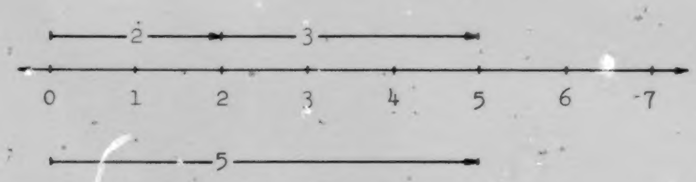
The operation of addition may be vividly pictured on the number line. Recall that the number line is constructed by placing marks on a line so that the segment between any two neighboring marks is congruent to one chosen segment. This was accomplished by laying off copies of the chosen segment end to end. The chosen segment determines a unit in the number line.



To visualize 2 + 3 = 5, let us first locate 2 and 5 on the number line; notice that between 2 and 5 are 3 units. Furthermore, we can observe that between 0 and 2 are 2 units.



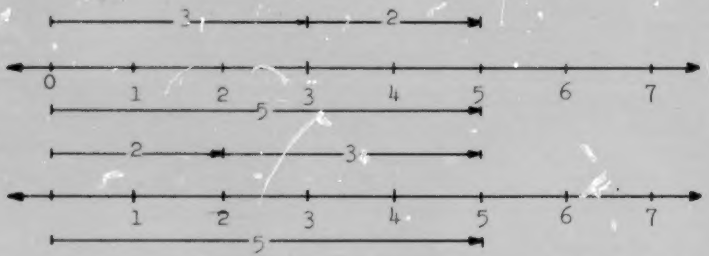
This process may be more effectively indicated by arrows as illustrated below, showing 2 + 3 = 5.



2 + 3 = 5

The above diagram shows an addition using the number line. More than this, however, the example may be interpreted also as an illustration of the closure property. An arrow of 2 units "followed by" an arrow of 3 units yields an arrow of a whole number of units. Each unit may be regarded as a step. Thus, 2 steps followed by 3 steps result in a total of 5 steps. Note that the steps originate from 0 as starting point and that we advance in accord with the increasing order of numbers.

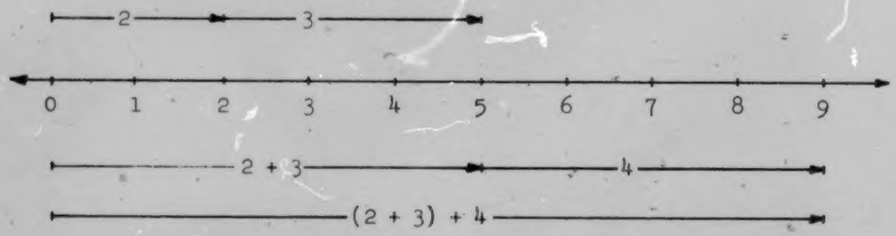
Consider now the sum $3 + 2$ on the number line. Here, 3 steps are followed by 2 steps and it is clear that we get the same result as before. Incorporating the diagrams for $3 + 2 = 5$ and $2 + 3 = 5$ into a single diagram, we can illustrate the commutative property under addition.



The associative property can also be illustrated using the number line. However, the process is more involved. As an example, we know that

$$(2 + 3) + 4 = 2 + (3 + 4).$$

The first expression, $(2 + 3) + 4$, may be illustrated by a simple extension of the above method. An arrow of 5 units results from the 2 and 3 unit arrows. To this, is abutted (attached end to end) the 4 unit arrow, thus



$$(2 + 3) + 4 = 9$$

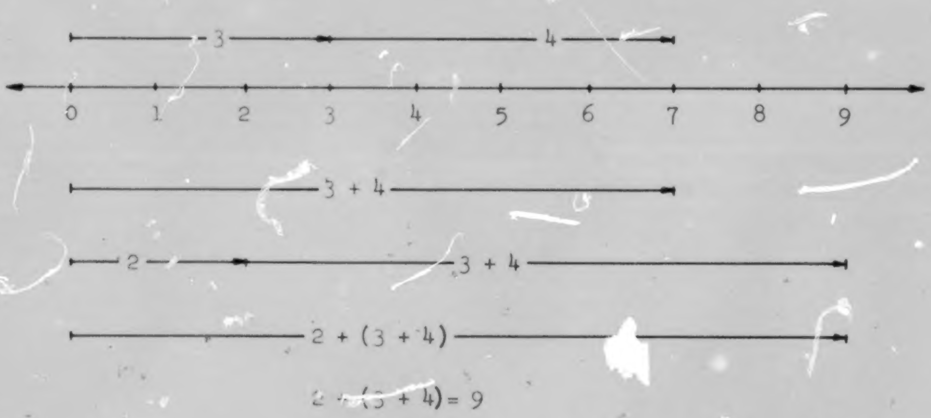
This of course, is analogous to the chain of statements

$$(2 + 3) + 4 = 5 + 4 = 9.$$

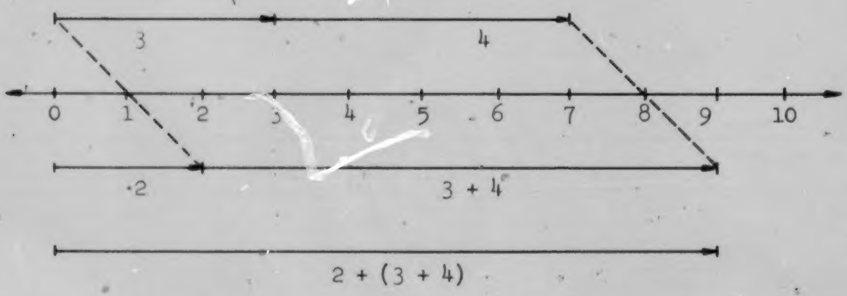
The illustration for the second expression, $2 + (3 + 4)$, is not as direct. For this, it may be more helpful to start with the analogous situation first. In analyzing $2 + (3 + 4)$, we note that $3 + 4 = 7$: that is "$3 + 4$" and "7" are names for the same number. Thus,

$$2 + (3 + 4) = 2 + 7 = 9 .$$

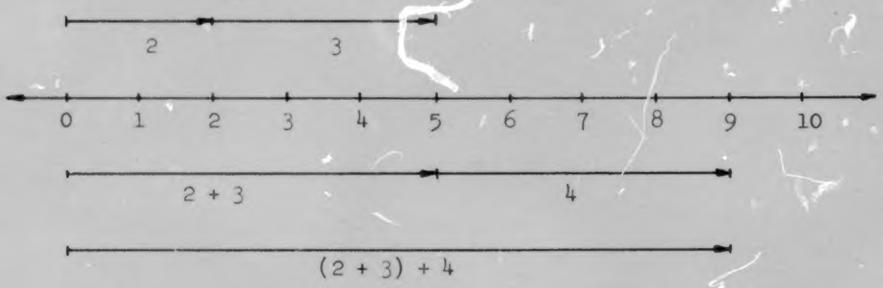
Accordingly, we are seeking an arrow corresponding to $3 + 4$. This arrow is then abutted to the arrow of 2 units to arrive at the result for $2 + (3 + 4)$.

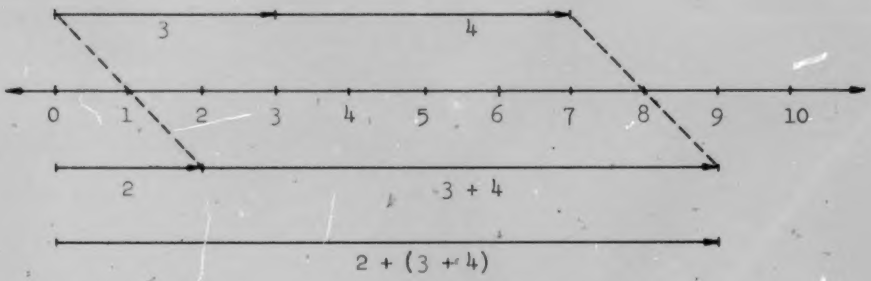

3
4
0
1
2
3
4
5
6
7
8
9
3 + 4
2
3 + 4
2 + (3 + 4)
2 + (3 + 4) = 9

The diagramming may be simplified by transferring the arrow for $3 + 4$ directly onto the 2 unit arrow as is shown below by the dotted lines:


3
4
0
1
2
3
4
5
6
7
8
9
10
2
3 + 4
2 + (3 + 4)

It is by incorporating the diagrams for $(2 + 3) + 4 = 9$ and for $2 + (3 + 4) = 9$ that we show associativity.





Frequent use of the number line to illustrate addition of whole numbers will promote familiarity with properties under addition. Thus the number line can help a great deal in working with numbers and in answering questions about numbers.

Problems

8. Draw number lines to show the following addition examples.
   - a. $3 + 6 = 9$
   - b. $4 + 5 = 9$
   - c. $(3 + 6) + 7 = 16$
   - d. $3 + (6 + 7) = 16$
9. Draw number lines to show that the following numbers are commutative under addition.
   - a. 3 and 5
   - b. 30 and 50
   - c. $(3 + 6)$ and 7

10. Are the diagrams in Problem 9c the same as those in Problems 8c and 8d? Why or why not?

11. How would arrows be used to indicate advancing from one point on the whole number line to the next point? What does this suggest about the whole number immediately following a given whole number a?

Number Property of the Product Set

When sets are disjoint, we have seen how the operation of addition may be related to the union of the sets. The sum of the number properties of all the sets is the number property of the union. Since multiplication may be viewed in terms of repeated addition, forming union after union would yield the number property required. For example, if we want the result of 4×5, we can get this by the union of 4 disjoint sets, each having 5 members.

$A = \{a,b,c,d,e\}$, $B = \{f,g,h,i,j\}$, $C = \{k,l,m,n,o\}$, $D = \{p,q,r,s,t\}$

Thus, $N(A \cup B) = 10$, $N((A \cup B) \cup C) = 15$, $N(((A \cup B) \cup C) \cup D) = 20$. This would call for finding 4 equivalent, but disjoint, sets. Another approach is by the use of the product set. This approach reveals more clearly how multiplication arises directly as an operation on two sets of numbers.

Using the same problem 4×5, that we have before, let us now consider two sets,

$$E = \{a,b,c,d\} \text{ and } F = \{e,f,g,h,i\},$$

then $N(E) = 4$ and $N(F) = 5$. The product set (Cartesian product) is

$$E \times F = \{(a,e),(a,f),(a,g),(a,h),(a,i),(b,e),(b,f),(b,g),(b,h),(b,i),$$
$$(c,e),(c,f),(c,g),(c,h),(c,i),(d,e),(d,f),(d,g),(d,h),(d,i)\}$$

from which $N(E \times F) = 20$. The Cartesian product of two sets thus gives directly the product of their number properties. If e is the number property of E, and f is the number property of F, then the number property of $E \times F$ is $e \times f$. In short,

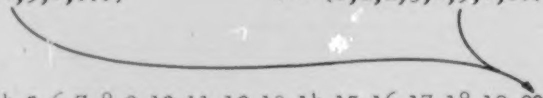
$$N(E) \times N(F) = N(E \times F).$$

It can be observed, moreover, that this statement is true whether or not the two sets are disjoint.

Rectangular Arrays

Via Cartesian products, multiplication is defined on sets of whole numbers. For example, $4 \times 5 = 20$, as the operation

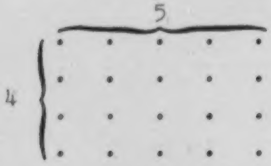
$W = \{0,1,2,3,4,5,6,\ldots\}$ $\qquad$ $W = \{0,1,2,3,4,5,6,\ldots\}$

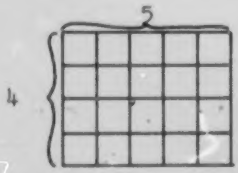


$W = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,\ldots\}$, can evolve from the product set $A \times B$, where $A = \{a,b,c,d\}$ and $B = \{\alpha, \beta, \gamma, \delta, \epsilon\}$. For these sets, $A \times B$ may be displayed as follows:

$$\begin{array}{llllll} A \times B = \{ & (a,\alpha), & (a,\beta), & (a,\gamma), & (a,\delta), & (a,\epsilon) \\ & (b,\alpha), & (b,\beta), & (b,\gamma), & (b,\delta), & (b,\epsilon) \\ & (c,\alpha), & (c,\beta), & (c,\gamma), & (c,\delta), & (c,\epsilon) \\ & (d,\alpha), & (d,\beta), & (d,\gamma), & (d,\delta), & (d,\epsilon)\} \end{array}$$

In this display, we can see that since $A \times B$ is the union of 4 equivalent disjoint sets, $\{(a,\alpha), (a,\beta), (a,\gamma), (a,\delta), (a,\epsilon)\}\{(b,a), (b,\beta), \ldots (b,\epsilon)\}, \ldots, \{(d,\alpha), (d,\beta), (d,\gamma), (d,\delta), (d,\epsilon)\}$, a rectangular array of 4 disjoint sets, each having 5 members would give us the number property 4×5. Thus, for a physical interpretation of 4×5, we may set up a rectangular array of 4 rows with 5 objects in each row. Counting the number of objects in the array gives the answer to 4×5. Either of the diagrams below, an array of dots or an array of rectangular shapes, can serve as a model for 4×5.





On the basis of such arrays, we can think of multiplication in terms of counting sets as follows:

> Given numbers a and b, an a by b rectangular array of objects can be constructed such that there are a rows and b columns in the array. The number, $a \times b$, is the number of objects in the array.

Problems

12. Using two sets that are not disjoint, one having 3 members and the other 4 members, show that the number property of the product set is 3×4.

13. a. Form a rectangular array of rectangular shapes illustrating an interpretation of 3×4.

    b. Using $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$ list the ordered pairs (a,b), where a is an element of A and b is an element of B within the rectangular shapes drawn above. Let a refer to the row and b to the column occupied by the rectangular shape.

14. Using the example 2×6, show by diagram how the multiplication table illustrates an operation on whole numbers, as was done for addition on page 151.

Properties under Multiplication

In the above, we have related multiplication to the product set. The result of the operation on any pair of numbers we call the product of the two numbers.

When we examined the union of two sets to get an insight into the properties under addition, we observed that the union of two sets is a set. The product set may similarly be examined to gather some information on the properties under multiplication. As in the case with the union, the product set of two sets is also a set. It is true that the elements of the product set are not elements of the original sets--they are pairs of elements. But, the crucial point is that the Cartesian product is a set, and a number property may be assigned to this set. From this, we have the closure property of whole numbers under multiplication:

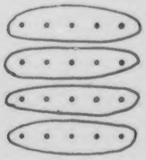
The product of two numbers is a whole number.

If $A = \{a,b,c,d\}$ and $B = \{\alpha, \beta, \gamma, \delta, \epsilon\}$, then the product set $A \times B$ is a set with 20 members. We have seen that if $A \neq B$, then the Cartesian product $B \times A$ is different from $A \times B$ since the pairs are ordered. For example, (a, β) is a member of $A \times B$ whereas (β, a) is a member of $B \times A$. By displaying the members of $B \times A$ as we had done for $A \times B$ we should see that $B \times A$ also has 20 members.

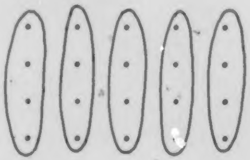
$$\begin{aligned} B \times A = \{&(\alpha,a), & (\alpha,b), && (\alpha,c), && (\alpha,d),\\ &(\beta,a), & (\beta,b), && (\beta,c), && (\beta,d),\\ &(\gamma,a), & (\gamma,b), && (\gamma,c), && (\gamma,d),\\ &(\delta,a), & (\delta,b), && (\delta,c), && (\delta,d),\\ &(\epsilon,a), & (\epsilon,b), && (\epsilon,c), && (\epsilon,d)\}. \end{aligned}$$

Therefore, even though $A \times B \neq B \times A$, both product sets are equivalent; that is, they have the same number property.

Notice from the above displays that an array of 5 disjoint sets each having 4 members, and an array of 4 disjoint sets, each having 5 members, have the same number property.



4 sets, 5 members
in each set



5 sets, 4 members
in each set

Since multiplication refers only to the number properties of sets involved in the Cartesian product, the fact that the Cartesian product is not commutative has no bearing on the commutativity under multiplication. It is still true that we have the *commutative property of whole numbers under multiplication*:

for any whole numbers *a* and b, $a \times b = b \times a$.

In the example that we have used, $4 \times 5 = 5 \times 4$. A 4 by 5 array has the same number of members as a 5 by 4 array. The array as a union of 4 disjoint sets, each having 5 members also shows that 4×5 can be computed by the successive addition.

$$\overbrace{5 + 5 + 5 + 5}^{4 \text{ addends}},$$

that is, 5 is used as an addend 4 times. (This is sometimes referred to as the repeated addition description of multiplication.)

Although multiplication of whole numbers may be described in terms of repeated addition, it must be remembered that multiplication is defined as an operation on two sets of numbers independent of addition. The operation showing the association of a third number with a given pair may be indicated, for example, by the usual method: $4 \times 5 = 20$ or simply $(4,5) \rightarrow 20$. "$(4,5) \rightarrow 20$" may be read: "to 4 and 5 is assigned the number, 20". Likewise, addition may be so described; thus $(4,5) \rightarrow 9$ may refer to an operation of addition.

Problems

15. Draw two arrays of rectangular shapes to illustrate that $3 \times 4 = 4 \times 3$.

16. Is it possible to draw an array to illustrate 3×0? Why or why not?

17. For each operation given below, state which arithmetic operation it refers to.

a. $(2,5) \rightarrow 10$	d. $(1,1) \rightarrow 1$
b. $(3,5) \rightarrow 8$	e. $(1,1) \rightarrow 2$
c. $(5,0) \rightarrow 5$	d. $(2,2) \rightarrow 4$

18. In adding, there is a particular number $\underline{a}$ such that $a + a = a$; find this number.

19. In multiplication, is there a number $\underline{a}$ such that $a \times a = a$? Are there more than one number $\underline{a}$ such that $a \times a = a$?

20. If possible, draw an array for $a \times a$ such that $a \times a = a$.

We have defined multiplication by the number property of the Cartesian product of two sets. There is no overt indication yet that the product of three or more numbers can be given directly by sets. If we want the product $3 \times 4 \times 5$, for example, what we might do is to find the Cartesian product of sets having 3 and 4 members each. This yields a set having 12 members. To find 12×5, we can use a set with 12 members and a set with 5 members, forming the product set of these two. This would be so provided we want the product, with the factors grouped: $(3 \times 4) \times 5$. For example, if

$$S = \{\bigstar, \square, \triangle\} \text{ and } E = \{a,b,c,d\}$$

then $S \times E$ is a set with 12 members:

$$S \times E = \{(\star,a),\ (\star,b),\ (\star,c),\ (\star,d)\ \ldots,\ (\triangle,d)\}.$$

Now, if D and G are sets with 12 and 5 members respectively, say,

$D = \{$dog, cat, horse, cow, goat, pig, chicken, duck,
sheep, goose, turkey, donkey$\}$

and

$$G = \{\alpha, \beta, \gamma, \delta, \epsilon\}$$

then a product set, $D \times G$ may be formed having 60 members.

Notice that D is equivalent to $S \times E$; there is a 1-1 correspondence between their members.

$$\begin{array}{lcccccccccccc}
D = \{ & \text{dog,} & \text{cat,} & \text{horse,} & \text{cow,} & \text{goat,} & \text{pig,} & \text{chicken,} & \text{duck,} & \text{sheep,} & \text{goose,} & \text{turkey,} & \text{donkey}\} \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
S{\times}E = \{ & (\star,a) & (\star,b) & (\star,c) & (\star,d) & (\square,a) & (\square,b) & (\square,c) & (\square,d) & (\triangle,a) & (\triangle,b), & (\triangle,c), & (\triangle,d)\}
\end{array}$$

Instead of $D \times G$, we might have used $(S \times E) \times G$ to find the number corresponding to $(3 \times 4) \times 5$. Then, some of the members of $(S \times E) \times G$ may be listed as follows:

$$(S \times E) \times G = \{((\star,a),\alpha),((\star,a),\beta),((\star,a),\gamma),\ldots,((\triangle,d),\epsilon)\}$$

Observe that each member of $(S \times E) \times G$ involves two sets of parentheses: the inner set specifies an ordered pair of $(S \times E)$ and the outer set specifies an ordered pair belonging to $(S \times E) \times G$ consisting of a member of $(S \times E)$ and a member of G. By agreeing that a member of a particular set always appears in the same position within the parentheses, we may be able to simplify the notation slightly. We might write a member of $(S \times E) \times G$ with the agreement that the first element within a set of parentheses is to be a member of S, the second element a member of E, and the third element, a member of G. Thus,

$$((\star,a),\alpha) \text{ might be simplified as } (\star,a,\alpha).$$

The simplification gives a triple of numbers; as with an ordered pair, such triples are ordered insofar as the order of listing elements within the parentheses must be observed. It is then possible to extend the concept of Cartesian products to ordered triples, quadruples, and so on.

In the foregoing, we examined the product set $(S \times E) \times G$. We can similarly examine the product set $S \times (E \times G)$ to find the product

$3 \times (4 \times 5)$. It is clear that if we do so, $E \times G$ will be revealed to have $4 \times 5 = 20$ members, and that $S \times (E \times G)$ will have $3 \times 20 = 60$ members. Recall that $(S \times E) \times G$ also has 60 members. Thus, both $S \times (E \times G)$ and $(S \times E) \times G$ yield the same number property. (In fact, while we have noted in Chapter 4 that the Cartesian product is not commutative, it can be shown that it is associative.) This parallels the case with the operation of addition; we have thus, the associative property of whole numbers under multiplication:

for whole numbers a, b, and c,

$$(a \times b) \times c = a \times (b \times c).$$

For the example we have above

$$(3 \times 4) \times 5 = 12 \times 5 = 60$$

and

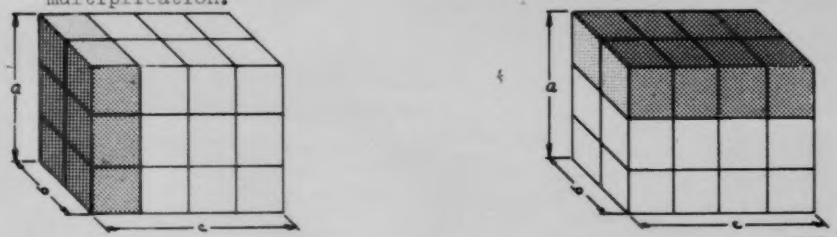
$$3 \times (4 \times 5) = 3 \times 20 = 60.$$

Alternately, this may be written as follows:

$$\begin{array}{ccc} (3 \times 4) \times 5 & & 3 \times (4 \times 5) \\ \shortparallel & & \shortparallel \\ 12 \times 5 & & 3 \times 20 \\ \shortparallel & & \shortparallel \\ 60 & = & 60 \end{array}$$

Showing again that $(3 \times 4) \times 5 = 3 \times (4 \times 5)$ by virtue of the statement, $60 = 60$; that is to say, both expressions name the same number.

The physical model of a box made up of cubical blocks with dimensions a by b by c, may be used to illustrate the associativity of multiplication.



$a \times b$ blocks in each vertical slice; c vertical slices.

$b \times c$ blocks in each horizontal slice; a horizontal slices.

Model illustrating the associative property of multiplication.

The number of blocks in such a box is $(a \times b) \times c$ and is also $a \times (b \times c)$ indicating that it is true that $(a \times b) \times c = a \times (b \times c)$.

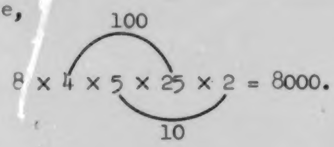
Problems

21. Show that $2 \times 3 \times 4 = 8 \times 3$ involves both the commutative and the associative properties of multiplication.

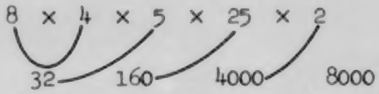
22. What property or properties are involved in each of the following?

a. $2 \times 3 \times 4 = 2 \times 12$
b. $2 \times 3 \times 4 = 3 \times 8$
c. $2 \times 3 \times 4 = 6 \times 4$
d. $2 \times 3 \times 4 = 2 \times 4 \times 3$
e. $2 \times 3 \times 4 = 3 \times 2 \times 4$
f. $4 \times 3 \times 2 = 4 \times 3 \times 2$

Each of the numbers 3, 4, or 5 in the product $3 \times 4 \times 5$ is called a factor of the product. The extensions of Cartesian products to more than two sets show that multiplication may be defined for more than two factors. Of course, this is implied by the closure property; since $a \times b$ is a whole number if a and b are whole numbers, we may proceed to find the product of $a \times b$ and c if c is a whole number. By the associative property, the product is unique, however the factors are grouped. Just as we could "pick and choose" pairs of addends in a sum, the commutative and associative properties under multiplication allow us to "pick and choose" pairs of factors in a product. For example,



Natural combinations yielding tens, hundreds, and so on might make for ease in computations. To be sure, for the same product, one can proceed to compute laboriously as follows:



Problem

23. Show by grouping with parentheses how $a \times b \times c \times d$ may be regarded as a product involving 3 factors instead of 4 for each of the following:

a. $2 \times 3 \times 4 \times 5 = 2 \times 3 \times 20$
b. $2 \times 3 \times 4 \times 5 = 6 \times 4 \times 5$
c. $2 \times 3 \times 4 \times 5 = 2 \times 12 \times 5$

The number 1 occupies, with respect to multiplication, the same position that 0 occupies with respect to addition. Notice that,

$$1 \times 3 = 3 \times 1 = 3,$$
$$1 \times 5 = 5 \times 1 = 5,$$
$$1 \times 6 = 6 \times 1 = 6,$$
$$1 \times 8 = 8 \times 1 = 8.$$

It is true that $1 \times a = a$ for all numbers *a* because a 1 by *a* array consists of only one row having *a* members, and therefore the entire array contains exactly *a* members.

$$1\{\overbrace{.\ .\ .\ .\ .}^{5}\} \qquad 1\{\overbrace{.\ .\ .\ .\ .\ .}^{6}\} \qquad 1\{\overbrace{.\ .\ .\ .\ .\ .\ .\ .}^{8}\}$$

$$1 \times 5 = 5 \qquad 1 \times 6 = 6 \qquad 1 \times 8 = 8$$

Since $1 \times a = a$, the number 1 is called the *identity* element for multiplication. The property is referred to as the *property of 1 under multiplication*:

for whole numbers *a*, $1 \times a = a$.

Because of the commutative property under multiplication, we also have $a \times 1 = a$.

While 0 does not act as the identity in multiplication, it does have a special role. The number of members in a 0 by 3 array (that is, an array with 0 rows, each have 3 members) is 0 because the set of members of this array is empty. In general, if *a* is a whole number, the number of members in a 0 by *a* array is 0; thus,

for whole numbers *a*, $0 \times a = 0$.

It is also true that $a \times 0 = 0$.

The characteristics of 0 in multiplication of "annihilating" (so to speak) all numbers except 0 in the product has an important consequence. If any factor is 0, the product is 0.

What has been done so far shows that multiplication, as with addition, is an operation on the whole numbers which has the properties of closure, commutativity and associativity. There is a special number 1 that is an identity for multiplication just as 0 is an identity for addition. Moreover, 0 plays a special role in multiplication for which there is no corresponding property in addition.

There is another important property that links the operations of addition and multiplication. This property which we shall now study is the basis, for example, for the following statement:

$$4 \times (7 + 2) = (4 \times 7) + (4 \times 2).$$

This example may be verified by noting that both $4 \times (7 + 2)$ and $(4 \times 7) + (4 \times 2)$ give the same result:

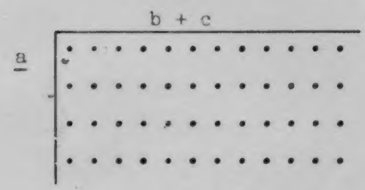
$$4 \times (7 + 2) = 4 \times 9 = 36, \text{ and}$$

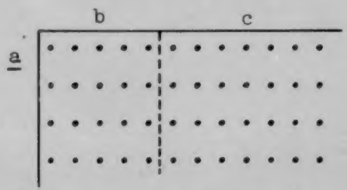
$$(4 \times 7) + (4 \times 2) = 28 + 8 = 36.$$

The property is called the distributive property of multiplication over addition. The distributive property states that if a, b and c are any whole numbers, then

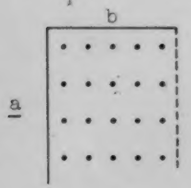
$$a \times (b + c) = (a \times b) + (a \times c) .$$

The distributive property may be illustrated by considering an a by $(b + c)$ array.

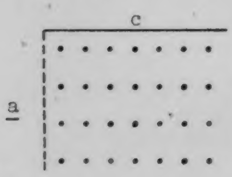




It is true that this array is formed from an a by b array and an a by c array.



An a by b array



An a by c array

Consequently, the number $a \times (b + c)$ of members in the large array is the sum of $(a \times b)$ and $(a \times c)$, the numbers of members of the subsets. That is, $a \times (b + c) = (a \times b) + (a \times c)$.

Since multiplication is commutative, both the "left hand" and the "right hand" distributive properties hold, that is,

Left hand: $a \times (b + c) = (a \times b) + (a \times c)$, and
Right hand: $(b + c) \times a = (b \times a) + (c \times a)$.

For example, by these distributive properties,

Left hand: $3 \times (5 + 8) = (3 \times 5) + (3 \times 8)$, and
Right hand: $(4 + 7) \times 2 = (4 \times 2) + (7 \times 2)$.

Recalling that when we say $A = B$ we mean A and B both name the same thing, then if $A = B$, it really makes no difference whether we write $A = B$ or $B = A$. With this in mind, since the left hand distributive property says that $a \times (b + c)$ and $(a \times b) + (a \times c)$ name the same number, the statement

$$a \times (b + c) = (a \times b) + (a \times c)$$

can equally well be written as

$$(a \times b) + (a \times c) = a \times (b + c).$$

For example,

$$(3 \times 5) + (3 \times 8) = 3 \times (5 + 8).$$

Similarly, the right hand distributive property may be expressed as either

$$(b + c) \times a = (b \times a) + (c \times a)$$

or

$$(b \times a) + (c \times a) = (b + c) \times a .$$

For example,

$$(4 \times 2) + (7 \times 2) = (4 + 7) \times 2.$$

The distributive property is very important as it is the basis for computing the product of two numbers.

$$\begin{aligned} \text{Left hand:} \quad (5 \times 4) + (5 \times 6) &= 5 \times (4 + 6) \\ &= 5 \times 10 = 50; \text{ also} \\ \text{Right hand:} \quad (7 \times 9) + (3 \times 9) &= (7 + 3) \times 9 \\ &= 10 \times 9 = 90. \end{aligned}$$

The convenience may be further illustrated by the following examples:

$$(9 \times 17) + (9 \times 83) = 9 \times (17 + 83) = 9 \times 100 = 900;$$
$$(24 \times 17) + (26 \times 17) = (24 + 26) \times 17 = 50 \times 17 = 850;$$
$$(854 \times 673) + (146 \times 673) = (854 + 146) \times 673 = 1000 \times 673 = 673{,}000;$$
$$(84 \times 367) + (84 \times 633) = 84 \times 1000 = 84{,}000.$$

Problems

24. Use the distributive property to compute each of the following:

    a. $(57 \times 7) + (57 \times 93)$

    b. $(57 \times 8) + (57 \times 93)$ [Hint: $8 = 7 + 1$]

25. Show that $(57 \times 5) + (57 \times 5) = 57 \times 10$ by the distributive property.

One might question whether addition distributes over multiplication. That is, is it always the case that

$$a + (b \times c) = (a + b) \times (a + c)?$$

This would be false if any set of numbers a, b and c can be found that would disprove the statement. For example, $a = 1$, $b = 3$, and $c = 2$ may be tried. For these values,

$$a + (b \times c) = 1 + (3 \times 2) = 1 + 6 = 7, \text{ but}$$
$$(a + b) \times (a + c) = (1 + 3) \times (1 + 2) = 4 \times 3 = 12 .$$

So it cannot be stated that $a + (b \times c)$ is always equal to $(a \times b) + (a \times c)$.

Summary of Properties

The properties of addition and multiplication developed so far for whole numbers may be summarized as follows, where a, b and c are whole numbers.

1. Whole numbers are CLOSED under addition and multiplication

   $a + b$ and $a \times b$ are whole numbers.

2. Addition and multiplication are COMMUTATIVE operations

   $a + b = b + a$ and $a \times b = b \times a$.

3. Addition and multiplication are ASSOCIATIVE operations

   $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

4. There is an IDENTITY element 0 for addition and an IDENTITY element 1 for multiplication

   $a + 0 = a$ and $a \times 1 = a$.

5. Multiplication is DISTRIBUTIVE over addition

   $a \times (b + c) = (a \times b) + (a \times c)$.

6. Zero has a special multiplication property

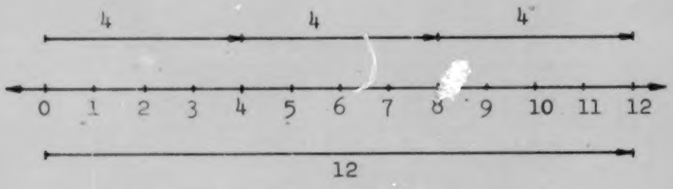
   $0 \times a = 0$.

Multiplication Using the Number Line

Through the interpretation of multiplication as repeated addition, multiplication may be illustrated on the number line. For example, 3×4 means 3 addends, each addend being 4. That is,

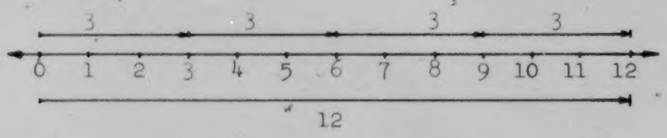
$$3 \times 4 = 4 + 4 + 4.$$

Therefore, this may be represented by 3 successive arrows as shown below:



$3 \times 4 = 12$

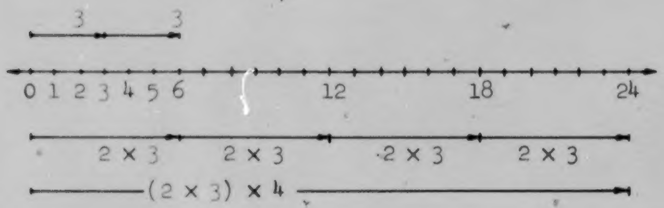
On the other hand, 4×3 means 4 addends of 3. The representation on the number line is as follows:



$$4 \times 3 = 12$$

As we can see, the two representations above are different; however, both of these yield the same result. By combining these two in a single diagram, we illustrate the commutative property under multiplication.

When more than two factors are involved, this too may be illustrated. For example, to show $(2 \times 3) \times 4$, we have the following.



$$(2 \times 3) \times 4 = 24$$

Likewise, $2 \times (3 \times 4)$ may be shown by obtaining two (3×4) "arrows" and abutting them. By combining the diagrams for $(2 \times 3) \times 4$ and $2 \times (3 \times 4)$, associativity may be illustrated.

Problem

26. Represent multiplication on the number line for $2 \times (3 \times 4)$.

Number Sentences

In developing the properties of numbers and various operations on numbers, we have been using a rather special language involving:

Symbols for numbers, such as: 1, 5, 2, 9, 3, ...;
Symbols for operations, such as: +, ×;
and Symbols showing relations between numbers,
such as: =, >, <.

A great deal of mathematics is in the form of sentences about numbers or number sentences as they are called. Sometimes the sentences make true statements as in "$9 + 5 = 14$", sometimes the number sentences are false as in "$5 + 7 = 11$". Whether it is true or false no more disqualifies the statement as a sentence than the statement, "George Washington was vice president under Abraham Lincoln" is a disqualified as a sentence.

Any number sentence has to have a "verb" or "verb form". The ones we have encountered so far are: "is equal to", "is less than", "is greater than". The symbols which we use for these verbs are listed below with a number sentence illustrating the use of each.

$=$; "is equal to"; $3 + 4 = 7$
$<$; "is less than"; $5 < 2 \times 5$
$>$; "is greater than"; $7 + 1 > 7$

As we have noted, verbal sentences may be true; "George Washington was the first President of the United States," or false: "Abraham Lincoln was the first President of the United States." We also encounter sentences such as: "He was the first President of the United States." If read out of context, it may not be known to whom "he" referred and it may thus be impossible to determine whether the sentence is true or false. In fact, "☐ was the first President of the United States" may be a test question requiring the name of the man for which it would be a true sentence. Such a sentence is called an open sentence and is of great usefulness not only in history tests but in many other situations as well. Open number sentences are the basis of a great deal of work in arithmetic. Solving a problem in arithmetic, for example, incorporates the notion of an open sentence. As an illustration, the problem

$\begin{array}{r} 7 \\ +\ 5 \\ \hline \end{array}$ may be stated: $7 + 5 = \square$ or $7 + 5 = \underline{\quad}$.

The number that makes $7 + 5 = \square$ a true statement is the solution for

$$\begin{array}{r} 7 \\ +\ 5 \\ \hline \end{array}.$$

Open number sentences are called equations if the verb in them is "=". Sentences with any of the other verbs listed above are called "inequalities".

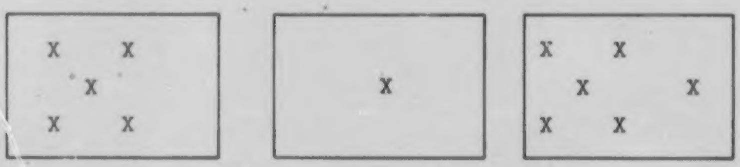
Problem

27. Write $<$, $>$, or $=$ in each blank so each mathematical sentence is true.

a. 8 ____ 6
b. 3 + 4 ____ 16
c. (20 + 30) ____ (30 + 20)
d. (200 + 800) ____ (200 + 700)
e. (1200 + 1000) ____ (1000 + 1200)

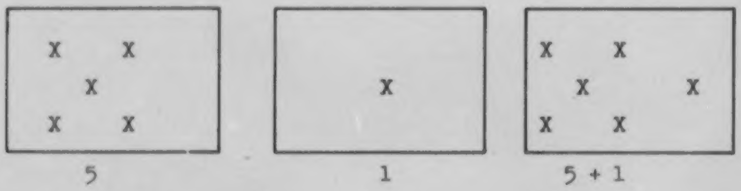
Applications to Teaching

Addition is associated with the union of disjoint sets of objects. By this, the commutative property is clearly illustrated; whether we join the first set to the second set or the second set to the first, the union consists of the same members. Recording results of joining sets using numerals may cause some difficulty without some intermediate steps. For example, from the diagram



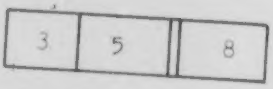
some children might not be able to proceed directly to the number sentence, 5 + 1 = 6.

A suggestion is to separate this problem into different tasks. Use of the flannel board to display objects in each set will be helpful. Then the numerals may be written below each picture with the numeral for the union showing the addends.



This may be followed by a review of the procedure the next day, writing 6 below 5 + 1 and finally, completion of the equation ______ + ______ = ______.

In forming their own sentences to accompany a pictorial situation, some children may have difficulty getting the "=" symbol in the right place. Drawing a double line between the appropriate frames may help with the association of ideas.



The use of the number line has been reported to be extremely helpful. A number line is fastened to each child's desk; the child eventually operates independent of this device in accord with his own rate of development.

Commutativity under multiplication may be conveyed by arranging chairs facing the board, for example, in an array of 10 rows, 2 to each row. When the chairs are turned 90° from the original direction, there will be 2 rows, 10 to each row. In each case (10 × 2 or 2 × 10), the number of children is 20.

The associative and distributive properties are not presented until the second grade. To illustrate the distributive property 4 sacks, each containing, say, 5 red blocks and 3 yellow blocks may be used. Thus, in the 4 sacks, there are 20 red blocks and 12 yellow blocks, or, 32 blocks.

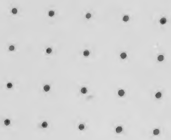
$$4 \times (5 + 3) = (4 \times 5) + (4 \times 3).$$

Exercises - Chapter 6

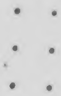
1. Show by trying to indicate the steps in repeated addition how the commutative property of multiplication would simplify the calculation of 1000 × 3.

2. What mathematical sentence is suggested by each of the arrays below?

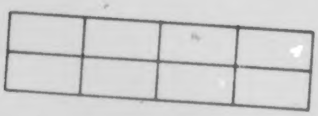
a.



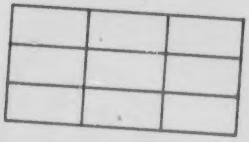
b.



c.



d.

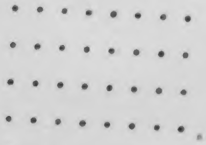


3. Mr. Rhodes is buying a two-tone car. The company offers tops in 5 colors and bodies in 3 colors. Draw an array that shows the various possible results, assuming that none of the body colors are the same as any of the top colors.

4. Mr. Rhodes is buying a two-tone car. Colors available for the top are: red, orange, yellow, green and blue. Colors available for the body are: red, yellow and blue. Draw an array to show the various possible results. If Mr. Rhodes insists that the car must be two-toned, how many choices does he have?

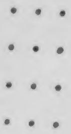
5. An ensemble of sweater and skirt is offered with the sweater available in five different colors and the skirt in 4 colors. The skirt also comes in either straight or flare style for each of the 4 colors. How many different ensembles are possible?

6. Here is an array separated into two smaller arrays.



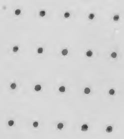
$(n = 4 \times 8)$

Array A



$(p = 4 \times 3)$

Array B



$(q = 4 \times 5)$

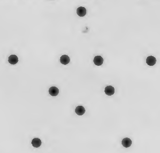
Array C

a. How many dots are in array A? Array B? Array C?

b. Does $n = p + q$?

c. Does $4 \times 8 = (4 \times 3) + (4 \times 5)$?

7. A familiar puzzle problem calls for planting 10 trees in an orchard so there are 5 rows with 4 trees in each row. The solution is in the form of the star shown in the figure to the right. Why doesn't this star illustrate the product of 5 and 4?



8. The middle section of an auditorium seats 28 to a row, and each side section seats 11 to a row. What is the capacity of this auditorium if there are 20 such rows?

9. What property of numbers is used in the following regrouping?

   $96 + 248 = 96 + 4 + 244 = 100 + 244 = 344.$

10. Use the commutative and associative properties to get the answer quickly by "picking and choosing" appropriate combinations:
    a. $5 \times 4 \times 3 \times 2 \times 1$
    b. $125 \times 7 \times 5 \times 8$
    c. $250 \times 14 \times 4 \times 2$

11. What does the following operation indicate for 3×4?

$$\begin{tikzcd}(3, 4) \arrow[dr] & \\ & 12 \\ (4, 3) \arrow[ur] & \end{tikzcd}$$

12. Make each of the following a true statement illustrating the distributive property.
    a. $3 \times (4 + \_\_) = (3 \times 4) + (3 \times 3)$
    b. $2 \times (\_\_ + 5) = (2 \times 4) + (\_\_ \times 5)$
    c. $13 \times (6 + 4) = (13 \times \_\_) + (13 \times \_\_$
    d. $(2 \times 7) + (3 \times \_\_) = (\_\_ + \_\_) \times 7$

13. a. If A is a set, give a proper subset B of A such that $N(A \cup B) = N(A) + N(B)$.
    b. How does the above reconcile with the concept of using disjoint sets as models for the sum?

Solutions for Problems

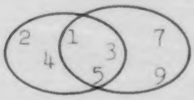
1. a. {1,2,3} d. {1,2,3,4,5,7,9}
   b. {1,2,3,4,6} e. {1,2,3,4,5,7,9}
   c. {1,2,3} f. {a,b,c,d,α,β,γ,δ,ε}

2.

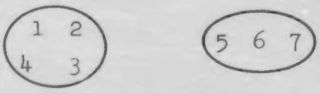
	a	b	c	d	e	f
N(A)	2	3	3	5	5	4
N(B)	1	3	0	5	5	5
N(A B)	3	5	3	7	7	9

N(A) + N(B) = N(A∪B) holds when A and B are disjoint; hence it holds in a., c., and f.

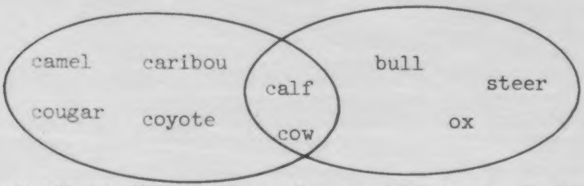
3. a.



b.



c.


camel
caribou
cougar
coyote
calf
cow
bull
steer
ox

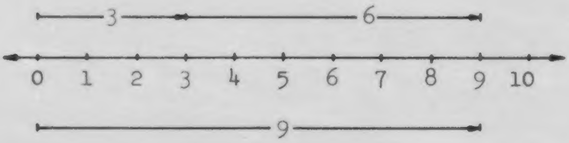
4. C is the set of possible numbers for an addend and B is the set of possible numbers for the sum.

5. a., c., f., and g.

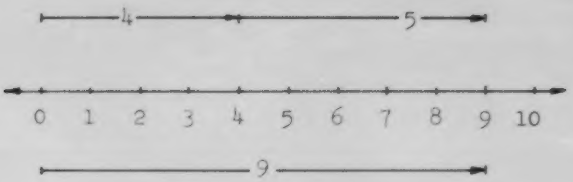
6. b., d., e., and g.

7. a. commutative property
   b. commutative property
   c. closure property
   d. commutative property
   e. no property; statement is false
   f. associative property
   g. commutative and associative properties.

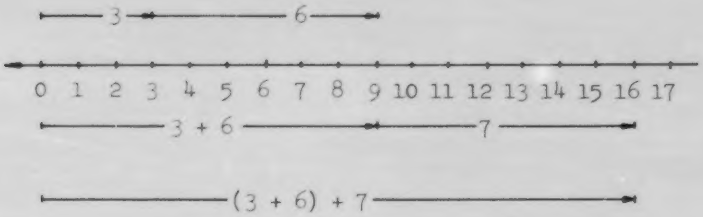
8. a.


3
6
0 1 2 3 4 5 6 7 8 9 10
9

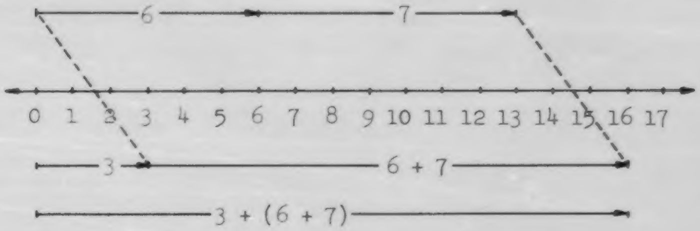
b.


0 1 2 3 4 5 6 7 8 9 10

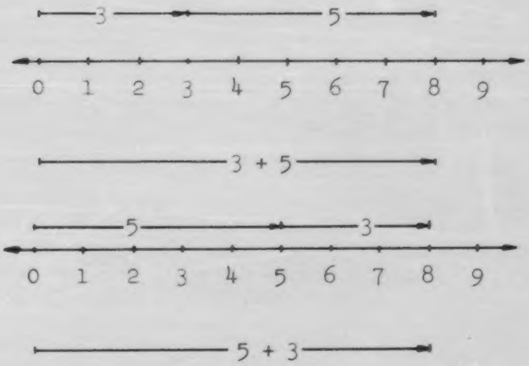
c.


3
6
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
3 + 6
7
(3 + 6) + 7

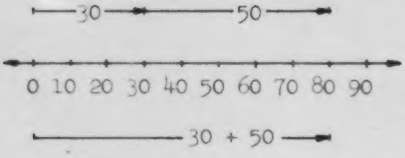
d.

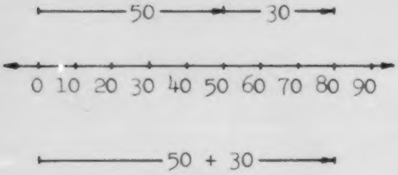

6
7
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
3
6 + 7
3 + (6 + 7)

9. a.

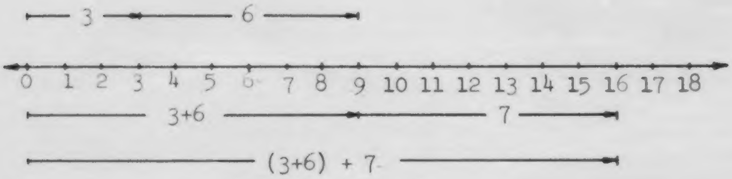

3
5
0 1 2 3 4 5 6 7 8 9
3 + 5
5
3
0 1 2 3 4 5 6 7 8 9
5 + 3

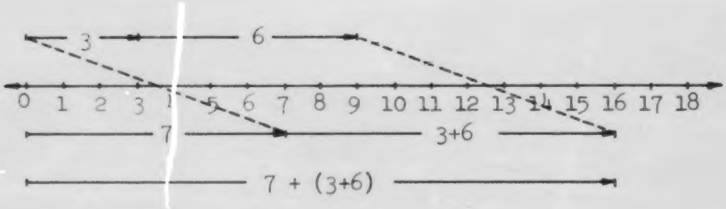
b.


30
50
0 10 20 30 40 50 60 70 80 90
30 + 50


50
30
0 10 20 30 40 50 60 70 80 90
50 + 30

c.


3
6
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
3+6
7
(3+6) + 7


3
6
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
7
3+6
7 + (3+6)

10. No; 8c and 8d show associativity of $3 + 6 + 7$; 9c shows commutativity of $(3 + 6)$ and 7.

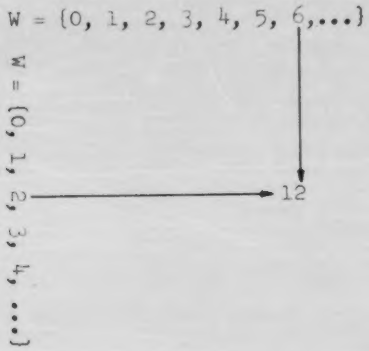
11. Abutting a 1 unit arrow to an arrow corresponding to a given number. This shows that the whole number after a given whole number a is obtained by adding 1 to a.

12. For example, if $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, then $A \times B = \{(a,a), (a,b), (a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d)\}$.

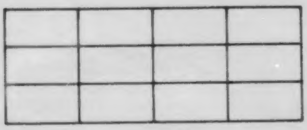
13.

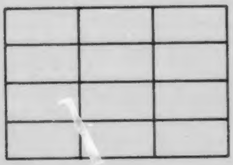
(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)

14.



15.





16. No; 3×0 is the number property of the empty set.

17.
- a. multiplication
- b. addition
- c. addition
- d. multiplication
- e. addition
- f. addition or multiplication

18. $a = 0$

19. Yes; either $a = 0$ or $a = 1$.

20.

21.

$2 \times 3 \times 4 = 2 \times (3 \times 4)$	associative property
$= 2 \times (4 \times 3)$	commutative property
$= (2 \times 4) \times 3$	associative property
$= 8 \times 3$	renaming

22.	a.	$2 \times 3 \times 4 = 2 \times (3 \times 4) = 2 \times 12$	associative
	b.	$2 \times 3 \times 4 = (2 \times 3) \times 4$	associative
		$= (3 \times 2) \times 4$	commutative
		$= 3 \times (2 \times 4)$	associative
		$= 3 \times 8$	
	c.	$2 \times 3 \times 4 = (2 \times 3) \times 4 = 6 \times 4$	associative
	d.	$2 \times 3 \times 4 = 2 \times 4 \times 3$	commutative
	e.	$2 \times 3 \times 4 = 3 \times 2 \times 4$	commutative
	f.	$4 \times 3 \times 2 = 4 \times 3 \times 2$	none involved

23. a. $2 \times 3 \times 4 \times 5 = 2 \times 3 \times (4 \times 5) = 2 \times 3 \times 20$

b. $2 \times 3 \times 4 \times 5 = (2 \times 3) \times 4 \times 5 = 6 \times 4 \times 5$

c. $2 \times 3 \times 4 \times 5 = 2 \times (3 \times 4) \times 5 = 2 \times 12 \times 5$

24. a. $(57 \times 7) + (57 \times 93) = 57 \times (7 + 93) = 57 \times 100 = 5700$

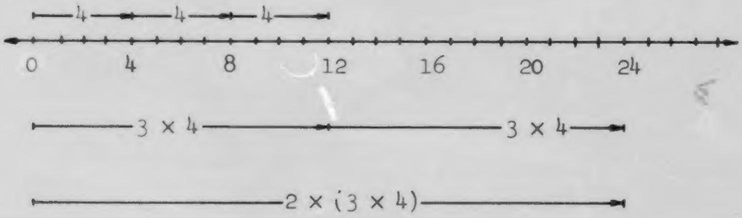
b. $(57 \times 8) + (57 \times 93) = (57 \times (1 + 7)) \times (57 + 93)$

$= (57 \times 1) + (57 \times 7) + (57 \times 93) = (57 \times 1) + (57 \times (7 + 93))$

$= (57 \times 1) + (57 \times 100) = 57 + 5700 = 5757$

25. $(57 \times 5) + (57 \times 5) = 57 \times (5 + 5) = 57 \times 10 = 570$

26.



27. a. >

b. <

c. =

d. >

e. =

Chapter 7

SUBTRACTION AND DIVISION

The Remaining Set

If A = {Cornelia, Sally, Jimmy, Emily, Elsie, Edward, Douglas} and if B = {Cornelia, Sally, Emily, Elsie}, then B is a subset of A. When B is specified as a subset of A, another subset of A is simultaneously specified; namely, by all the elements of A that are *not* elements of B. In this way, an operation is defined, producing from A and B, a set called the *complement of B* relative to A, or more simply, the *remaining set*. Thus, if C = {Jimmy, Edward, Douglas}, and A and B are as above, then C is the remaining set.

Together, the union of B and C is A, so the two subsets "complete" the given set. Since C is composed of elements that are not elements of B, it is clear that the intersection of B and C is the empty set. In fact, these last two statements can be used as the basis for defining the relative complement, or remaining set. We denote the operation by the symbol "~", read "wiggle". For example, if A = {0, △, □, , ☽} and B = {0, □}, then $A \sim B$ = {△, ★, ☽}. Of course, the goal is to connect this operation with subtraction, and this goal is immediately achieved by looking at the appropriate number properties. Note that in this example, the number property of A is 5, the number property of B is 2, and the number property of $A \sim B$ is 3. In general, it is true that

$$N(A \sim B) = N(A) - N(B).$$

Since the definition of $A \sim B$ requires B to be a subset of A, there are evidently restrictions on B. B can be the empty set; B can be identical to A; these two sets, A and the empty set, establish the limits on B. Consequently, if $N(A) = a$ and $N(B) = b$, we have the restrictions $b \geq 0$ and $b \leq a$. (The symbol "$\geq$" combines "$>$" and "$=$" to indicate "is greater than *or* equal to"; similarly "$\leq$" is read "is less than *or* equal to". The restrictions can be incorporated into the one statement, $0 \leq b \leq a$; that is, the number of elements in B can range from 0 to the number of elements in A. These limitations for subtraction are eventually relaxed when the set of numbers that we have to work with is extended to include more than just the whole numbers. The pattern of development proceeds thus: from observations on complementation, the characteristics of subtraction are examined; from examination of the characteristics, the operation is extended. As a result, numbers other than whole numbers may be introduced. For example,

$$\text{if } A = \{a, b, c, d, e\} \text{ and}$$
$$B = \{a, b, c\}, \text{ then } A \sim B = \{d, e\}.$$

From this, we get the difference

$$N(A) - N(B) = N(A \sim B); \text{ that is}$$
$$5 - 3 = 2.$$

The statement, $5 - 3 = 2$, may in turn trigger the question whether subtraction may be defined for any two whole numbers. For example, is $5 - 8$ defined? If we limit ourselves to the set of whole numbers, the answer is "no". But by reassessing the behavior of subtraction, it is possible to introduce new members to the number system so that subtraction is always defined in the system.

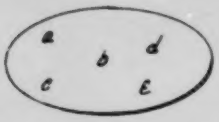
The example, $5 - 8$, brings out two important features of the subtraction operation. Since no whole number is the result of $5 - 8$, the set of whole numbers is not closed under subtraction. Contrasted with $8 - 5$, which does yield a whole number for an answer, we see that in general, if a and b are whole numbers, it is not true that $a - b$ is the same as $b - a$. Thus, subtraction is neither closed nor commutative. These are negative results; they tell us some of the properties that subtraction does not have. Nevertheless, these are important results.

Subtraction as Inverse

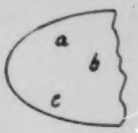
Subtraction is not restricted to only negative results, however; nor is the operation of getting remaining sets so restricted. A noteworthy result may be stated thus:

$$(A \sim B) \cup B = A .$$

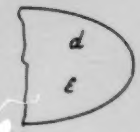
In words: If we form the remaining set $A \sim B$, and then form the union of it with B, we have the original set, A. Diagrammatically, the situation may be illustrated as follows:



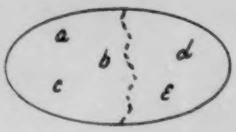
A



B



$A \sim B$



$(A \sim B) \cup B$

Similarly, if we start out with a set, X, and join a disjoint set Y to it, we get $X \cup Y$. Now if we take the complement of Y relative to $X \cup Y$, then we have $(X \cup Y) \sim Y$, which turns out to be X, the original set. That is,

$$(X \cup Y) \sim Y = X.$$

Because of these two situations, we say that the union and the complementation are inverse operations. In effect, one operation "undoes" what is done by the other. Corresponding to these properties under the set operations, we have similar properties under addition and subtraction:

if a and b are whole numbers, and
$b \leq a$, then $(a - b) + b = a$ and
$(a + b) - b = a$.

Therefore, subtraction and addition are inverse operations whenever the two operations are possible or defined.

Definitions of Subtraction

We have defined the difference as the number property of the remaining set. This gives us a means of finding $a - b$ if a is a number and if b is a number less than or equal to a. We first choose a set, A, such that $N(A) = a$; next we pick a set, B, which is a subset of A and such that $N(B) = b$. These two sets determine the remaining set, $A \sim B$. The number, $a - b$, is the number of elements in $A \sim B$:

$$a - b = N(A \sim B).$$

For example, if $a = 5$ and $b = 2$, we can choose A to be the set

$$A = \{O, \triangle, \square, \star, \mathcal{E}\}.$$

Next we can choose B to be the subset

$$B = \{\triangle, \star\}.$$

Then

$$A \sim B = \{0, \square, \mathcal{E}\}.$$

Now our definition tells us that

$$5 - 2 = N(A \sim B) = 3.$$

Note that if we made a different choice for B, for example

$$B = \{\square, \mathcal{E}\},$$

the result would be the same. Also, if we had chosen a different set, A, for example $A = \{V, W, X, Y, Z\}$, and any two member subset of this set as B, the result would still be the same.

Problem*

1. Use this definition of subtraction to compute in detail $7 - 3$.

There is a second approach to subtraction which does not use the idea of the remaining set, but uses the ideas of union of disjoint sets and of one-to-one correspondence. If a is a number and if b is a number with $b \leq a$, we start by choosing a set A with $N(A) = a$ and a set B disjoint from A with $N(B) = b$.

Next we choose a set C, disjoint from both A and B in such a way that A and $(B \cup C)$ are in one-to-one correspondence. That is, there is a pairing of the elements of A with the elements of $B \cup C$. Then the second definition of subtraction is:

$$a - b = N(C).$$

In other words, having chosen appropriate disjoint sets A and B we look for a third set C with just the right number of members so that the union of this set and the set B will exactly match up with the set A. The number of members in such a set C tells us "how much larger" A is than B.

As an example of this definition of subtraction let us again use $a = 5$ and $b = 2$. A can be the same set $\{0, \triangle, \square, \bigstar, \mathcal{E}\}$ as was used before, but B must now be a disjoint set with 2 members. Let $B = \{X, Y\}$. An attempt to get a one-to-one correspondence between the elements of B and the elements of A may result in the following,

---

*Solutions for problems in this chapter are on page 208.

$$\begin{array}{rcccccl} B = \{ & X, & Y \} & & & & \\ & \updownarrow & \updownarrow & & & & \\ A = \{ & O, & \triangle, & \square, & \star, & \mathcal{E} & \}, \end{array}$$

leaving some elements of A unpaired. We look for a set, C (disjoint from B) so that $B \cup C$ will match A. Thus, if $C = \{\alpha, \beta, \delta\}$, then the elements of $B \cup C$ can be put into one-to-one correspondence with those of A.

$$\begin{array}{rcccccl} B \cup C = \{ & X, & Y, & \alpha, & \beta, & \delta & \} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ A = \{ & O, & \triangle, & \square, & \star, & \mathcal{E} & \}. \end{array}$$

Now by the second definition of subtraction, the result of $5 - 2$ is the number property of C. Therefore, $5 - 2 = N(C) = 3$. The most important thing to say about this definition of subtraction is that it always gives exactly the same result as the first definition.

Problem

2. Use the second definition of subtraction to compute in detail $7 - 3$.

Now the question naturally arises as to why we should bother with two different definitions if they both give the same result. Why not use just one of them?

The reason is that there are two quite different kinds of problems that we commonly meet and it is important to know that the same mathematical operation can be used to solve both kinds of problems.

The first kind is the "take away" type:

> "John has 5 dollars and loses two of them. How many dollars does he have left?"

The second kind is the "how many more" type:

> "John has 5 dollars. Bill has 2 dollars. How many more dollars does Bill need in order to have as many as John?"

The first definition of subtraction fits very well with the "take away" type of problem, and the second fits very well with the "how many more" type. But in each case the problem is solved by means of the subtraction: $5 - 2 = 3$.

The statement that we have on the preceding page, relating addition to subtraction, namely

$$(a - b) + b = a,$$

gives us yet another insight into the concept of subtraction. If $a - b$ is some number c, then we have

$$c + b = a.$$

In other words, $a - b$ is that number c such that $a = c + b$. This is why we can say that

$$a - b = c \text{ if and only if } a = c + b;$$

these two statements mean exactly the same thing.

From this point of view, subtraction is defined as the operation of finding the unknown addend, c, in the addition problem

$$a = c + b$$

since this is the same number as $a - b$. For example, we can state that $5 - 2$ is $\underline{3}$ because $5 = \underline{3} + 2$.

Also, since we know that both

$$5 = 3 + 2 \text{ and } 5 = 2 + 3$$

it is true that

$$5 - 2 = 3 \text{ and } 5 - 3 = 2.$$

In general, any addition fact gives us two subtraction facts automatically.

Problems

3. The two statements $a - b = c$ and $a = c + b$ mean the same thing. Working with whole numbers 6, 4, and 2 show the related addition and subtraction facts.
4. When would it be that an addition fact does not give us two subtraction facts automatically?

There are two reasons why it is important for teachers to understand this way of thinking about subtraction, as well as the first two. The first is that this is the way that children usually think when they are developing their skills in computation. The second is that as children move through school, and study other kinds of numbers, such as fractions, decimals, negative numbers, etc., they will meet this idea of defining subtraction in terms of addition again and again.

It is important to realize that all three definitions of subtraction are equivalent and yield the same properties.

Properties under Subtraction

We have noted a property of subtraction that points to its role as an inverse of addition. Two properties of the whole numbers under this operation that we want to highlight now involve the empty set. Recall that with the union, we have

$$A \cup \{\ \} = A.$$

The corresponding statement for numbers is for any whole number a,

$$a + 0 = a.$$

By the above, we observe that

$$a + 0 = a \quad \text{and} \quad a = a - 0$$

say the same thing. Since $a + 0 = 0 + a$, we also have $0 + a = a$, which is the same as $0 = a - a$. Hence, in addition to the inverse properties,

for any whole numbers a and b, with $a \geq b$, $(a - b) + b = a$,

for any whole numbers a and b, $(a + b) - b = a$,

we have the following two properties of zero under subtraction:

for any whole number a, $a - 0 = a$;

for any whole number a, $a - a = 0$.

Problems

5. By a definition of subtraction, we see that $a - b = c$ if and only if $a = c + b$, and that $(a - b) + b = a$. Which properties are ememplified by the following?
   a. $(202 - 200) + 200 = 202$
   b. $(y - x) + x = y$
   c. $[(30 - 15) - 5] + 5 = 15$
   d. $5 + 0 = 5$
   e. $5 - 0 = 5$

6. Does the sentence $(5 - 7) + 7 = 5$ make sense for whole numbers?

7. Show by the use of the properties of addition and subtraction that the following sentence is true:

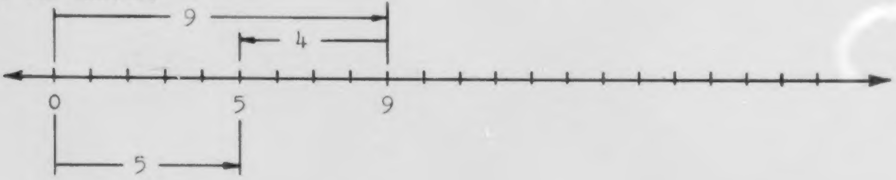
$$\text{If } b \geq a, \quad a + (b - a) = b.$$

Check that it is true by using several pairs of numbers.

Subtraction Using the Number Line

If we consider subtraction with respect to the representation of numbers using the number line, we can illustrate many of its important processes and properties.

What is the answer to $9 - 4$? We start on the number line at 9 and take away or move to the left 4 units thus arriving at 5, which is our answer.



$$9 - 4 = 5.$$

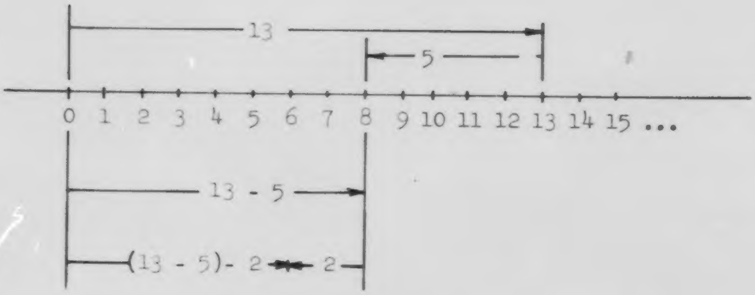
In Chapter 6 we illustrated the use of the number line to show the associative property of addition. Subtraction does not have the associative property for

$$(13 - 5) - 2 = 8 - 2 = 6$$

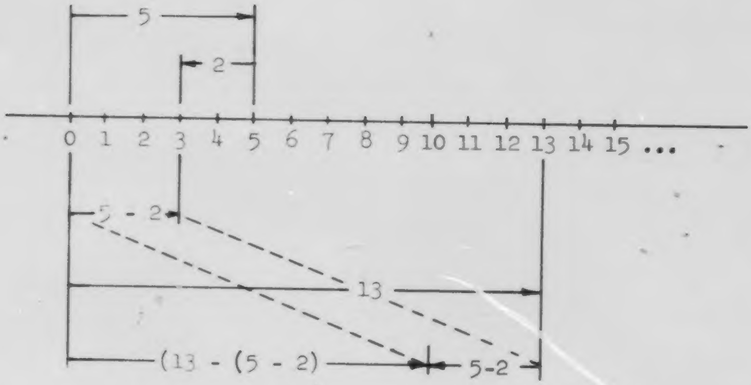
while

$$13 - (5 - 2) = 13 - 3 = 10.$$

These examples are illustrated on number lines below. The first figure shows that $13 - 5 = 8$, and this result is used to get 6 from $8 - 2$. The second shows that $5 - 2 = 3$ and this result is used to get 10 from $13 - 3$.


13
5
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
13 - 5
(13 - 5)- 2
2

$$(13 - 5) - 2 = 8 - 2 = 6$$


5
2
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 ...
5 - 2
13
(13 - (5 - 2)
5-2

$$13 - (5 - 2) = 13 - 3 = 10$$

Hence, it is not true that $(13 - 5) - 2$ is the same as $13 - (5 - 2)$, and we express this by the number sentnece.

$$(13 - 5) - 2 \neq 13 - (5 - 2),$$

where the symbol "$\neq$" means "is not equal to".

Division

In the preceding chapter, a rectangular array of a rows with b members in each row was used as a physical model for $a \times b$. From this and from other models, the properties of multiplication for whole numbers were developed. We saw that multiplication of whole numbers has the properties of closure, commutativity and associativity, and that multiplication is distributive over addition. Also, the numbers 1 and 0 have the special properties that

$$1 \times a = a \times 1 = a, \text{ and}$$

$$0 \times a = a \times 0 = 0.$$

The first three properties exactly parallel the same three properties for addition, and 1 plays a role for multiplication closely corresponding to that of 0 for addition. The similarity in behavior of the two operations leads to the question as to whether there is an operation which bears to multiplication a similar relation as subtraction does to addition; namely, an inverse or undoing operation. The answer to this is the operation called division.

To find the product 4×5, we counted the number of members in a 4 by 5 array or in 4 disjoint sets with 5 members in each set. An associated problem is to start with 20 objects and ask how many disjoint subsets there are in this set if each subset is to have 4 members. In terms of arrays, the question is "if a set of 20 members is arranged 4 to a row, how many rows will there be?" The answer is 5.

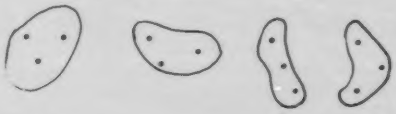
x x x x
x x x x
x x x x
x x x x
x x x x

20 objects arranged 4 to a row.

In many cases there would be no answer to the question, depending on the numbers. For example, 20 objects arranged 6 to a row does not give an exact number of rows. It is true that ordinarily we do carry out such a division process as 20 divided by 6, obtaining a quotient and a remainder. In speaking of division as an operation in the set of whole numbers, however, the expression "20 divided by 6" is meaningless because it is not a whole number. The process as indicated by $\begin{array}{r} 3 \\ 6\overline{)20} \end{array}$, remainder 2, will be more fully developed later when the techniques of division are discussed in detail. It will then be pointed out that for any ordered pair (a, b) with $b \neq 0$, we may develop a division process.

To answer the question, "how many disjoint subsets are there in a set of 20 if each subset is to have 4 members?", we formed an array of 20 objects arranged 4 to a row. When we form this array, we are partitioning the set of 20 into equivalent sets. By partitioning a set, we mean separating it into disjoint subsets. Thus, the fact that a set of 20 may be partitioned into 5 equivalent subsets, each having 4 members, shows us that $20 = 4 \times 5$ and $20 = 5 \times 4$. The number, 5, which is thus assigned to the ordered pair (20, 4) is called the quotient and the operation which produces 5 from (20, 4) is called division. The normal symbol for the operation of division is $\div$. Thus $20 \div 4 = 5$. The partitioning, of course, does not have to be shown as an array. Either diagram below, for example, gives the result of $12 \div 3$.

12 objects, 3 in each row.



Set of 12 objects in disjoint subsets, 3 objects in each subset.

For the ordered pair (20, 6) there is no such number that can be attached; nor is there for (5, 15). So, under the operation of division, (20, 6) or (5, 15) are not defined in the set of whole numbers. Division therefore does not have the property of closure in the set of whole numbers. The last case for (5, 15) is simply an example of the fact that in the ordered pair of whole numbers (a, b), if $b > a$, and $a \neq 0$, the operation of division never yields a whole number.

Problems

8. Find the whole number attached to each of the following ordered pairs under the operation of division; if there is none, explain.

   a. (20, 5)
   b. (4, 28)
   c. (6, 1)
   d. (72, 9)
   e. (64, 8)
   f. (42, 7)
   g. (47, 7)

9. a. Display an array to show $28 \div 7$.

   b. Illustrate $28 \div 7$ by a partitioning that is other than an array.

By partitioning, we have obtained 5 as the result of $20 \div 4$ because $20 = 5 \times 4$. This is similar to the missing addend approach to subtraction. Here, we say that $a \div b$ is that number c such that $a = c \times b$. That is,

$$a \div b = c \text{ if and only if } a = c \times b.$$

Thus, c is the missing factor of $a = c \times b$ for given numbers $\underline{a}$ and b, with $b \neq 0$.

Division as Inverse

In the same way as subtraction is the inverse of addition, division by a number n may be thought of as the inverse of multiplication by n. Thus,

$$(8 \times 3) \div 3 = 8 \text{ and } (17 \times 4) \div 4 = 17 .$$

However, caution must be exercised in thinking about multiplication as the inverse of division because it is true that

$$(15 \div 3) \times 3 = 15, \text{ while } (8 \div 3) \times 3 \text{ is meaningless}$$

since $8 \div 3$ is not a whole number. This is similar to the caution we must exercise in this "doing and undoing" process with subtraction; thus while

$$(15 - 3) + 3 = 15 \text{ is perfectly acceptable,}$$

$$(5 - 13) + 13 \text{ is meaningless}$$

since $(5 - 13)$ is not a whole number. Of course, the restriction will be removed later when the set of whole numbers is extended to include numbers for which $8 \div 3$ and $5 - 13$ have meaning.

Problems

10. Tell whether each of the following statements is true or whether it is meaningless for whole numbers.

a. $(3 + 9) - 9 = 3$
b. $(9 + 3) - 9 = 3$
c. $(3 - 9) + 9 = 3$
d. $(3 \times 9) \div 3 = 9$
e. $(3 \div 9) \times 9 = 3$
f. $(9 \times 3) \div 3 = 9$
g. $(9 \div 3) \times 3 = 9$

The Role of 1 and 0 in Division

The operation of division was connected to the operation of multiplication by the statement that

$$a \div b = c \text{ if and only if } a = c \times b.$$

Since 1 and 0 played special roles in multiplication, it may be appropriate to pay particular attention to the two numbers in division.

If $b = 1$, then we have $a \div 1 = c$ if and only if $a = c \times 1$. Recalling the special property of 1 under multiplication, we have $c \times 1 = c$; hence, a and c represent the same number, and for any whole number a, $a \div 1 = a$. On the other hand, $1 \div b$ is not a whole number unless $b = 1$; there is no whole number c such that $1 = c \times b$ if $b \neq 1$.

In the sense that $a \div 1 = a$, the number 1 acts somewhat like an identity element for division. Unlike the identity element for multiplication in which, for any a, $1 \times a = a \times 1$, the number 1 is limited to acting as an identity element for division only if it is to the right of the symbol ÷ .

Again by the definition of division, we can note the role of 0 in division. Briefly, its role may be summarized as follows.

$0 \div b = c$ if and only if $0 = c \times b$. For $b \neq 0$, this is true only if $c = 0$. Therefore,

for any whole number b such that $b \neq 0$, $0 \div b = 0$.

If $b = 0$, we have $0 \div 0 = c$ and $0 = c \times 0$. Since this is true for any number c, the result of $0 \div 0$ is ambiguous; $0 \div 0$ does not specify a unique number, hence

the operation of division is not defined for $0 \div 0$.

$a \div 0$ where $a \neq 0$ is still another situation. Since $a \div 0 = c$ if and only if $a = c \times 0$, and $c \times 0 = 0$ for whatever number c, we have a contradiction in terms; we started out with the assumption that $a \neq 0$ and came to the conclusion that $a = 0$. For this reason,

for $a \neq 0$, $a \div 0$ is undefined.

These last two results together indicate that division by 0 is not defined.

Problems

11. Tell whether each of the following is a whole number, is not a whole number, or cannot be determined; if possible, name the whole number.

a. $8 \div 4$
b. $2 \div 4$
c. $3 \div 3$
d. $6 \div 0$
e. $0 \div 132$
f. $1 \div b$, b is a whole number and $b = 1$.

g. $1 \div b$, b is a whole number and $b \neq 1$.

h. $a \div b$, a and b are whole numbers and $b > a$.

i. $0 \div b$, b is a whole number and $b \neq 0$.

j. $a \div b$, a and b are whole numbers and $a > b$.

k. $a \div b$, a and b are whole numbers and $a = b$.

Properties of Division

Many examples may be given to show that the whole numbers are not closed under division. For example, while $6 \div 3 = 2$, $3 \div 6$ is not a whole number. These same two examples show that $6 \div 3 \neq 3 \div 6$, hence the operation is not commutative. To see that division is not associative, again many examples may be produced. We need only one example, and such an example is the following:

$$(12 \div 6) \div 2 = 2 \div 2 = 1, \text{ but}$$

$$12 \div (6 \div 2) = 12 \div 3 = 4.$$

The different results obtained for $(12 \div 6) \div 2$ on the one hand, and for $12 \div (6 \div 2)$ on the other, shows that, in general, it is not true that $(a \div b) \div c = a \div (b \div c)$.

So far, division with respect to whole numbers has revealed itself as an operation that does not have the properties of closure, commutativity and associativity. Furthermore, division by 0 is to be avoided. To free ourselves from the impression that not much can be said about this operation, we need to consider only the important notion that division by b is the inverse of the operation of multiplication by b. That is, $(a \times b) \div b = a$, provided, of course, $b \neq 0$.

Problems

12. For which of the following is it true that $(a \div b) \div c = a \div (b \div c)$?

    a. $4 \div 2 \div 2$

    b. $4 \div 2 \div 1$

    c. $24 \div 6 \div 2$

    d. $0 \div 5 \div 1$

    e. $9 \div 9 \div 1$

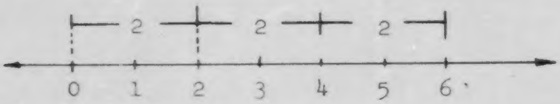
    f. $9 \div 3 \div 1$

    g. $0 \div 9 \div 3$

13. From the results of the preceding exercises, under what conditions will $(a \div b) \div c = a \div (b \div c)$?

Division Using the Number Line

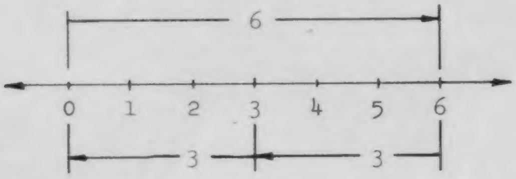
We can illustrate division using the number line by partitioning a segment into congruent subsegments. For example, to illustrate $6 \div 3$, we can partition a 6 unit segment into 3 congruent subsegments, each of which



is congruent to the segment from 0 to 2. Thus, this partition conveys the concept $6 \div 3 = 2$. Clearly, this is associated with the representation of multiplication on the line in which three 2 unit arrows or 2 unit segments are abutted, resulting in a 6 unit arrow or a 6 unit segment. The association may be thought of as: one operation is the inverse of the other, or, from the point of view that

$$6 \div 3 = 2 \text{ if and only if } 6 = 2 \times 3.$$

Another method of illustrating division on the number line is related to considering division in terms of repeated subtraction. This concept will be discussed in further detail in Chapter 13 when the division techniques are discussed. We can indicate here, however, this use of the number line in order to compare with the use shown above. Beginning with 6,



we ask: How many times can 3 be subtracted? Corresponding to this, we can show division using the number line as in the above figure. In this case, since subtraction is performed twice, $6 \div 3 = 2$.

Problems

14. a. Show by partitioning a segment on the number line that $10 \div 2 = 5$.

    b. Show by partitioning a segment on the number line that $5 \div 2$ does not yield a whole number.

Composite Numbers

Rectangular arrays form the basis for what used to be known as the "rectangular numbers" by the ancient Greeks. If a number n can be presented as other than a 1 by n array, then the n is said to be a rectangular number. For example, 6 may be represented by a 2 by 3 array, so 6 is a rectangular number. Now we call such a number a composite number; $6 = 2 \times 3$, so 6 is "composed" of 2 and 3. 12 is also a composite number; either a 3 by 4 rectangular array or a 2 by 6 rectangular array may be used as a model for the composition of 12. However, $2 \times 2 \times 3$ also shows how 12 may be composed. It is true that if a whole number n may be "decomposed" into more than two factors (other than 1 and n), then it can be decomposed into two factors other than 1 and n. Hence, such a number would be considered also a rectangular number. It is simply that thinking in terms of the composition puts the focus more on analyzing the number than thinking in terms of rectangular arrays that can be formed.

Since $12 = 3 \times 4$, we have regarded 3 and 4 as factors of 12. As we have noted, there are other factors of 12. For example, 2 is a factor of 12 because there is a whole number whose product with 2 is 12. That is, 2 is a factor of 12 because 12 is 2 times a whole number; in this case, the whole number is 6. This automatically qualifies 6 to be also a factor of 12. A complete list of factors of 12 may be catalogued as follows:

$12 = 1 \times 12$, so 1 and 12 are factors of 12;
$12 = 2 \times 6$, so 2 and 6 are factors of 12;
$12 = 3 \times 4$, so 3 and 4 are factors of 12;
$12 = 4 \times 3$, so 4 and 3 are factors of 12;
$12 = 6 \times 2$, so 6 and 2 are factors of 12;
$12 = 12 \times 1$, so 12 and 1 are factors of 12;

Thus, 12 has 1, 2, 3, 4, 6, and 12 as factors. 5 is not a factor of 12 because there is no whole number n such that the mathematical sentence

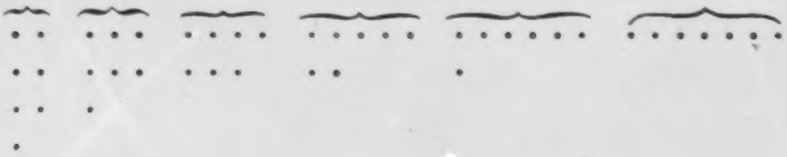
$$12 = 5 \times n$$

is true. Neither are 7,8,9,10,11, and any whole number greater than 12 factors of 12. (Notice that the last three statements in the display give no information on factors that was not contained in the first three statements and we could have done without them.)

It is clear that since $n = 1 \times n$, any whole number n has 1 and n as factors. However, there are many whole numbers for which these are the *only* factors. For example 1 and 5 are the only factors of 5; 1 and 7 are the only factors of 7; and 1 and 13 are the only factors of 13; and so on. Such numbers will be of interest for us and are specially designated.

> Any whole number that has *exactly* *two* different whole number factors (namely itself and 1) is a *prime* *number*.

Note that this definition excludes 1 from the set of prime numbers because 1 does not have two *different* factors. It also excludes 0 from the set of primes since $0 = 0 \times n$ for any whole number n; any whole number is a factor of 0. In essence, the prime numbers are those that can only be associated with a 1 by n array (for $n \neq 1$). For example, let us consider an array for 7. Placing two objects in each row, we can complete an array with 6 objects; the seventh object makes the array incomplete. Similarly,



3, 4, 5, or 6 objects in a row induce incomplete arrays with 7 objects.

All whole numbers greater than 1 may now be classified according to whether they are prime or composite. Over 2,000 years ago, the mathematician Eratosthenes devised an easy and straightforward method for sorting prime numbers from a list of whole numbers. To find all the prime numbers less than 50, for example, the whole numbers from 0 through 49 are listed as below. 0 and 1 are crossed out since they are not primes. 2 is a prime, but every other even number has 2 as a factor, so all even numbers greater than 2 are crossed out.

~~0~~,	~~1~~,	2,	3,	~~4~~,	5,	~~6~~,	7,	~~8~~,	9,
~~10~~,	11,	~~12~~,	13,	~~14~~,	15,	~~16~~,	17,	~~18~~,	19,
~~20~~,	21,	~~22~~,	23,	~~24~~,	25,	~~26~~,	27,	~~28~~,	29,
~~30~~,	31,	~~32~~,	33,	~~34~~,	35,	~~36~~,	37,	~~38~~,	39,
~~40~~,	41,	~~42~~,	43,	~~44~~,	45,	~~46~~,	47,	~~48~~,	49

Continuing with this, 3 is "saved" and 3 × 2, 3 × 3, 3 × 4, ..., are "eliminated"; that is, all "multiples" of 3 greater than 3 × 1 are eliminated.

0,	1,	②,	③,	4,	5,	6,	7,	8,	~~9~~,
10,	11,	12,	13,	14,	~~15~~,	16,	17,	18,	19,
20,	~~21~~,	22,	23,	24,	25,	26,	~~27~~,	28,	29,
30,	31,	32,	~~33~~,	34,	35,	36,	37,	38,	~~39~~,
40,	41,	42,	43,	44,	~~45~~,	46,	47,	48,	49

In this second chart the numerals that are shaded represent numbers that are "eliminated" after the screening as "multiples" of 2 (1 is "eliminated" before this screening). The slash marks indicate screening as "multiples" of 3, and the numbers that are "saved" are identified by circles. By now, 4 has been eliminated because it is a multiple of 2; 5 is next saved and all other multiples of 5 eliminated and so on. Thus, eventually, we arrive at the set of all prime numbers less than 50:

{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

It can be shown that this screening process needs not be carried beyond 7 for prime numbers less than 50 since 49 = 7 × 7. If 49 is the product of two whole numbers a and b, and one of these is greater than 7, then the other must be less than 7. This tells us that any factor greater than 7 would have been eliminated when its companion factor (which is less than 7) was considered.

Problems

14. Express each of the following numbers as products of two factors in several ways, or indicate that it is impossible to do so.

    a. 18

    b. 6

    c. 30

    d. 11

15. List all the numbers that could be called "factors"

  a. of the number 30,

  b. of the number 19,

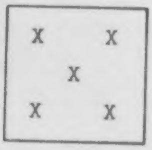
  c. of the number 24.

Applications to Teaching

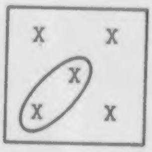
Some children find it difficult to visualize set removal. For them partitioning and ringing a subset is not enough; they cannot seem to appreciate that the objects have been removed since the objects are still much in evidence. Covering up the objects to be removed or crossing them out with an X may help communicate removal. Similarly, using a cup to cover up a subset of beans, for example, has been found to be effective in teaching set removal.

On the other hand, removal may have been so convincing that it causes difficulty with writing the number sentence associated with the removal. For example, in trying to connect the expression 5 - 2 with 3, only the numbers for the original set and the remaining set may be recorded; the other subset has been removed, so the child cannot understand why its number must be recorded. In that case, intermediate stages in the removing process may be suggest[illegible] This may be in the form of a class activity, for example, with a set of beans. The number of the set may first be recorded; a subset may next be separated, counted, and the number recorded. Removal may be accomplished by covering the set removed (as with a cup) and finally, the number in the remaining set identified and recorded.

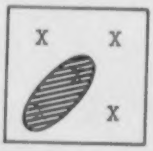
Intermediate stages for the recording of numbers in the ringing of set members may also be provided. For example, the following suggests various possible stages for 5 - 2 = 3.



5



5 - 2



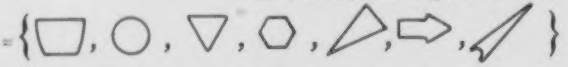
3

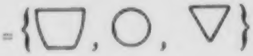
The concept of inverse may prove difficult. For this, a variety of examples may be required showing situations which have inverses such as falling asleep and waking up, say, or putting on a coat and taking it off. However, sometimes it is not the lack of understanding of the concept that is causing difficulty; it may be trying to verbalize the "doing and undoing" that the children find difficult.

The topics of factors, composite numbers, and prime numbers will not be presented until the second grade. A start on this is given in the first grade when odd numbers and even numbers are discussed. Of course, in terms of multiples, the even numbers are simply the multiples of 2. Similarly, multiples of 3 are the entries in the 3 times table, and so on.

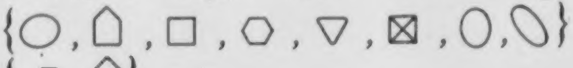
We have noted that since 3 is a factor of 12, we can say that 12 is a multiple of 3. Both factor and multiple originate from the same concept: there is a whole number n such that $12 = 3 \times n$. A multiple is viewed from the standpoint of the number being composed; a factor is viewed from the standpoint of a number going into the composition as a "building block". Beginning in Grade 5, the children will be introduced to the Fundamental Theorem of Arithmetic - when a whole number is "decomposed" into the primitive building blocks of prime numbers, this decomposition will be revealed as unique; that is, a whole number is made up of one and only one set of primitive blocks which we call the primes. At that time, the children will be taught the "complete factorization" of a whole number (or, the prime decomposition). Complete factorization is a natural lead-in to a corresponding factorization in algebra, which yields, among other things, solutions to algebraic equations.

Exercises - Chapter 7

1. A = 

   B = 

   Join to B a set C disjoint from B such that $B \cup C = A$.

2. If A = 

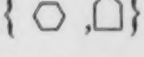
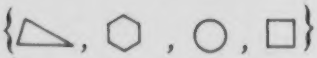
   and B = 

   exhibit $A \sim B$.

3. If from a set of 8 members we remove a set of 2 members, how many members does the resulting set have?

4. If A = 

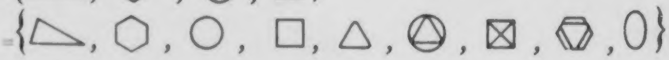
and C = 

exhibit B such that $A \cup B = C$. What is $N(B)$?

5. Show a representation on the number line which illustrates the fact that $10 - 3 = 7$. Use the same figure to illustrate the idea that $10 = 7 + 3$.

6. Show a representation on the number line which illustrates that the associative property does not hold under the operation of subtraction.

$$(9 - 6) - 3 \neq 9 - (6 - 3)$$

7. What operation is the inverse of adding 7 to any number? What is the inverse of subtracting 8?

8. If A and B are disjoint, illustrate that $(A \cup B) \sim B = A$. What happens if A and B are not disjoint?

9. Rewrite each mathematical sentence below as a division sentence. Find the unknown factor.

a. $n \times 5 = 20$
b. $p \times 4 = 28$
c. $n \times 1 = 6$
d. $n \times 9 = 72$
e. $n \times 8 = 64$
f. $q \times 0 = 0$

10. Tell whether each of the following is more readily visualized by a rectangular array of 7 rows or by disjoint subsets with 7 in each subset.

a. 42 pieces of candy are to be divided equally among 7 children.
b. 42 pieces of candy are to be packaged 7 pieces to a package.

11. A marching band always forms an array when it marches. The leader likes to use many different formations. Aside from the leader, the band has 59 members. The leader is trying very hard to find one more member. Why?

12. Does division have the commutative property? Give an example to substantiate your answer.

13. Express each of the following numbers as a product of two smaller numbers or indicate that it is impossible to do this:

a. 12
b. 36
c. 31
d. 7
e. 8
f. 11
g. 35
h. 5
i. 39
j. 42
k. 6
l. 41
m. 82
n. 95

Solutions for Problems

1. Choose $A = \{0, \triangle, \square, \star, \bigcirc, \mathcal{E}, \boxdot\}$ with $N(A) = 7$.
   Choose $B = \{\star, \bigcirc, \square\}$ which is a subset of A and $N(B) = 3$.
   $A \sim B = \{\square, 0, \mathcal{E}, \triangle\}$.
   By definition, we know that $7 - 3 = N(A \sim B) = 4$.

2. Choose $A = \{0, \triangle, \square, \star, \bigcirc, \mathcal{E}, \boxdot\}$ with $N(A) = 7$.
   Choose $B = \{a, b, c\}$ with $N(B) = 3$.
   Now choose a set C disjoint from both A and B.
   $C = \{\Theta, \Xi, \circledast, \Phi\}$ and $N(C) = 4$
   so that by matching $(B \cup C)$ with A we can put $B \cup C$ in one-to-one correspondence with A.

$$\begin{array}{rcccccccc} B \cup C = & \{a, & b, & c, & \Theta, & \Xi, & \circledast, & \Phi\} \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ A = & \{0, & \triangle, & \square, & \star, & \bigcirc, & \mathcal{E}, & \boxdot\} \end{array}$$

   By definition we know that $7 - 3 = N(C) = 4$.

3. By using whole numbers 6, 4, we can illustrate the fact that $a - b = c$ and $a = c + b$ mean the same thing. Thus

$$6 - 4 = 2 \text{ because } 6 = 2 + 4$$
and
$$6 - 2 = 4 \text{ because } 6 = 4 + 2.$$

4. When $a = b$, then $a + b = c$ gives only one subtraction fact; namely $a = c - b$. For example, $3 + 3 = 6$ and $3 = 6 - 3$.

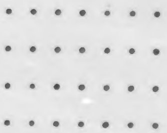
5. a. Inverse property of addition and subtraction
   b. inverse property of addition and subtraction
   c. inverse property of addition and subtraction showing grouping within the parentheses. $30 - 15$ is another name for 15.
   d. identity property of zero for addition (Zero added to any number results in that number.)
   e. identity property of zero for subtraction (Zero subtracted from any number results in that number.)

6. $(5 - 7) + 7$ does not make sense in the present context because $5 - 7$ is not a whole number. For any numbers a and b, $(a - b) + b = a$ if $a \geq b$.

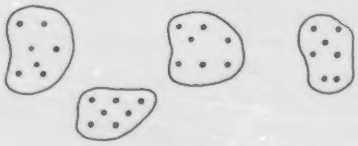
7. To show that $a + (b - a) = b$ if $b > a$ we use the commutative property of addition getting $a + (b - a) = (b - a) + a$, which by the third item in Properties of Subtraction is equal to b.

8. a. 4
   b. None; $28 > 4$ and $4 \neq 0$.
   c. 6
   d. 8
   e. 8
   f. 6
   g. None; there is no 7 row array of 47 members.

9. a.



b.



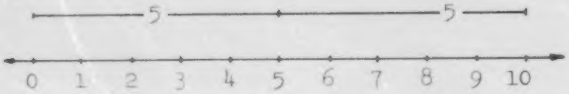
10. a. True
    b. True
    c. Meaningless
    d. True
    e. Meaningless
    f. True
    g. True

11. a. Whole number; 2
    b. Not a whole number
    c. Whole number; 1
    d. Not a whole number
    e. Whole number; 0
    f. Whole number; 1
    g. Cannot be determined: meaningless if $b = 0$; not a whole number if $b > 1$.
    h. Cannot be determined: zero if $a = 0$; not a whole number if $a \neq 0$.
    i. Whole number; 0
    j. Cannot be determined: meaningless if $b = 0$; whole number $\underline{a}$ if $b = 1$; whole number if $b > 1$ and b is a factor of $\underline{a}$; not a whole number if $b > 1$ and b is not a factor of a.
    k. Cannot be determined; undefined if $a = b = 0$; the whole number 1 if $a = b \neq 0$.

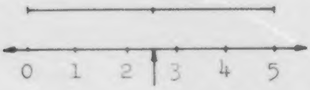
12. a. False
    b. True
    c. False
    d. True
    e. True
    f. True
    g. True

13. If $a = 0$, or $c = 1$, or both $a = 0$ and $c = 1$.

14. a.



b.



The coordinate of this point is not a whole number.

15. a. 3×6, 2×9, 1×18 (or 6×3, 9×2, etc.)

b. 2×3, 1×6

c. 2×15, 5×6, 3×10, 1×30

d. 1×11 and 11×1 are the only such factorizations and they are not essentially different.

16. a. 1, 2, 3, 5, 6, 10, 15, and 30

In more formal terms, the set of factors of $30 = \{1,2,3,5,6,10,15,30\}$

b. 1 and 19

c. The set of factors of $24 = \{1,2,3,4,6,8,12,24\}$

Chapter 8

NUMERATION: NAMING NUMBERS

Introduction

We have used whole numbers extensively in our work thus far. We have considered their nature, the nature of operations associated with them, and some properties of these operations. However, we have not considered explicitly the important distinction between numbers and their names. Now we turn our attention to this distinction and particularly to schemes for naming whole numbers; that is, to the problem of numeration.

Whole Numbers and Their Names

We know that the whole number "twelve", for example, is a property of the set

$$\{a, b, c, d, e, f, g, h, i, j, k, l\}$$

and of all sets equivalent to this set. The word "twelve" is a name for this number property and is not the number itself. Similarly, the symbol or numeral "12" is another name for this same number. This is true also for the numeral "XII" , written in the Roman system of notation. In fact, when we write

$$XII = 12$$

we simply are asserting that "XII" and "12" are two different names for the same thing; that is, the same number.

As we now consider principles of numeration, it is important for us to keep clearly in mind that number and numeral are not synonymous. A number is a concept, an abstraction. A whole number is one kind of number, and in various preceding chapters we have considered selected aspects of the whole number system. On the other hand, a numeration system is a system for naming numbers; thus, it is a numeral system. In this chapter, we shall be concerned with numeration systems for naming whole numbers. Our emphasis will be on the number names or numerals, rather than on the numbers themselves.

Earlier Numeration Systems

Man, during the course of his history, did not always use our familiar Hindu-Arabic numeration system. His earliest schemes involved little more than tally marks, such as / for "one", // for "two", /// for "three", etc. Such primitive schemes were far from effective and efficient, particularly when dealing with large numbers.

The Egyptians, the Chinese, the Greeks, the Romans, and others all developed numeration systems that were improvements upon primitive tally schemes. However, none of these was as sophisticated as the one developed by the Hindus, which evolved into the Hindu-Arabic system we use today. Nevertheless, a brief consideration of at least one of these earlier numeration systems can be of interest and can give us an appreciation of the principles and advantages of our own system.

A Modified Greek System

The Greek system of numeration used twenty-seven basic symbols: the twenty-four letters of the Greek alphabet and three obsolete letters. Each of these basic symbols named a particular number. Other numbers were named by combining basic symbols according to established principles or "rules".

Let us illustrate a modified version of this Greek system by using as basic symbols the twenty-six letters of our own alphabet and one additional arbitrary symbol, ▽ . The number named by each basic symbol is indicated below in terms of our own Hindu-Arabic numerals.

A = 1
B = 2
C = 3
D = 4
E = 5
F = 6
G = 7
H = 8
I = 9

J = 10
K = 20
L = 30
M = 40
N = 50
O = 60
P = 70
Q = 80
R = 90

S = 100
T = 200
U = 300
V = 400
W = 500
X = 600
Y = 700
Z = 800
▽ = 900

A compound symbol such as "PD" is interpreted to mean

$$70 + 4 \text{ , or } 74$$

in our own system. Similarly,

"WKH" means $500 + 20 + 8$, or 528 ,
"TR" means $200 + 90$, or 290 , and
"UF" means $300 + 6$, or 306

in terms of our familiar numerals.

Notice that the symbol "DP" would be interpreted to mean $4 + 70$, or 74 . Thus, it would be true that

$$PD = DP .$$

However, we shall agree that in such instances we shall write the basic symbol for the larger number to the left of the basic symbol for the smaller number. Thus, the preferred form would be PD instead of DP . Similarly, it would be true that

$$WKH = WHK = KHW = KWH = HWK = HKW .$$

Of these six different names for the same number, the preferred form would be WKH .

Problems[*]

1. Express each of these modified Greek system numerals as familiar Hindu-Arabic numerals.

   a. MG b. ZNB c. XK d. VC e. ▽RI

2. Express each of these Hindu-Arabic numerals in the "preferred form" of modified Greek system numerals.

   a. 63 b. 735 c. 210 d. 504 e. 888

3. Does the modified Greek system have a basic symbol for the number "zero"? If so, what is that symbol? If not, why is such a basic symbol not used in the system?

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[*] Solutions for problems in this chapter are on page 234.

But what about naming numbers greater than ▽RI , or 999? We cannot name such numbers without some further agreement or extension of the system. So, let us agree that we may use a prime mark (') to indicate that the number named by a basic symbol is to be multiplied by one thousand (1000) . Thus,

E' means 1000 × 5 , or 5000 ,
P' means 1000 × 70 , or 70,000 ,
and T' means 1000 × 200 , or 200,000

in terms of our familiar numerals.

Problems

4. Express each of these modified Greek system numerals as familiar Hindu-Arabic numerals.

a. B'YMG b. Q'A'UL c. V'O'RC

You undoubtedly have noticed that the number "ten" is of particular significance in the modified Greek numeration system. For instance, the symbols J, K, . . . , Q, R named multiples of ten (10, 20, . . . , 80, 90) , and the symbols S, T, . . . , Z, ▽ named multiples of ten tens or one hundred (100, 200, . . . , 800, 900) .

We may say that "ten" is the base of this numeration system. It is the basic number that we use for groupings within the system.

Features of Numeration Systems

Many numeration systems have three features that are of significance as we turn to a consideration of our own Hindu-Arabic system.

1. One of these features is that of base, a basic number in terms of which we effect groupings within the system. This number may or may not be "ten". If the base is "ten", we often refer to that system as a decimal system. ("Decimal" is derived from the Latin word decem which means "ten".)

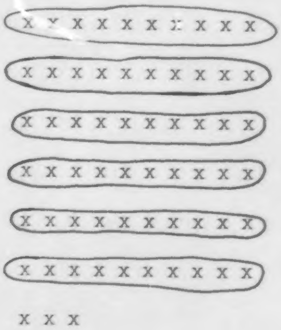
2. Another feature is a set of basic symbols or number names. From these, all other numerals are built. As we shall see, the choice of base often determines the number of basic symbols used within a numeration system.

3. A third feature is a set of principles or rules for combining basic symbols to form other numerals so that every whole number may be named in terms of these basic symbols only. It is within this third feature that we find a principle that sets the Hindu-Arabic system apart from others that preceded it. We are referring, of course, to the principle of place value.

The Hindu-Arabic Numeration System

Let us examine each of the preceding features as it relates specifically to our Hindu-Arabic numeration system.

1. The Hindu-Arabic numeration system is a decimal system: its base is ten. This is seen clearly in the fact that we interpret the number "sixty-three", for example, as "six tens and three (ones)". "Sixty" itself means "six tens". This feature may be illustrated in the groupings below for the interpretation of the number "sixty-three".

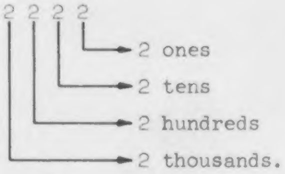


2. The Hindu-Arabic numeration system utilizes ten basic symbols or digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 such that

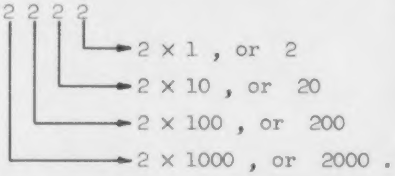
0 names the number zero;
1 names the number one;
2 names the number two;
3 names the number three;
4 names the number four;
5 names the number five;
6 names the number six;
7 names the number seven;
8 names the number eight;
and 9 names the number nine.

Notice the inclusion of a symbol for zero: 0 . This is in marked contrast to systems such as the Greek, the Roman, etc., that had no zero symbol. The need for a zero symbol in the case of the Hindu-Arabic system is related closely to the place value principle discussed in the following section.

3. The Hindu-Arabic numeration system utilizes a principle of place value, along with multiplicative and additive principles, in order to combine basic symbols or digits of the system to name whole numbers greater than nine. We are quite familiar with the fact that in the numeral 2222 , for instance, each digit 2 does not have the same "value". The "value" of each 2 is determined by its place or position in the numeral as a whole:



Or, we may convey the same idea in a slightly different way:

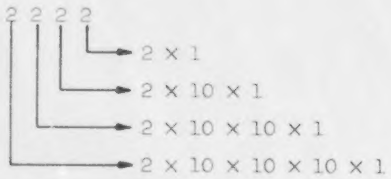


Here we see the multiplicative principle in association with the place - value principle.

We frequently find it helpful to use an expanded form of notation to emphasize both the multiplicative and additive principles that apply to the interpretation of a numeral such as 2222 :

$$2222 = (2 \times 1000) + (2 \times 100) + (2 \times 10) + (2 \times 1) .$$

None of the notations used thus far has made explicit the important role of the base, ten, in determining the "place values". Each place to the left of the ones place in a numeral has associated with it a "value" that is ten times the "value" associated with the place immediately to its right. For the numeral 2222 , we can show this important idea in this way:



or

$$2222 = (2 \times 10 \times 10 \times 10) + (2 \times 10 \times 10) + (2 \times 10) + (2 \times 1) .$$

The importance of the zero symbol, 0 , in connection with our place - value numeration system is reflected in numerals such as 2220 , 2202 , 2022 , 2200 , and 2002 . Without the zero symbol such numerals could not be distinguished readily from 222 (in the case of 2220 , 2202 and 2022) or from 22 (in the case of 2200 and 2002). Without some symbol to denote "not any" in a particular place, a numeration system with a place value principle would not be feasible. In fact, the relatively late invention of a symbol for "not any" (a symbol for the number pertaining to the empty set), was the reason for the relatively late creation of a place - value numeration system.

The following chart may be helpful in summarizing some of the ideas just discussed regarding our numeration system.

Millions			Thousands			Units		
	tens	ones	hundreds	tens	ones	hundreds	tens	ones
	10,000,000	1,000,000	100,000	10,000	1,000	100	10	1
	$10 \times \underline{10 \times 10 \times 10 \times 10 \times 10 \times 10}$	$10 \times \underline{10 \times 10 \times 10 \times 10 \times 10}$	$10 \times \underline{10 \times 10 \times 10 \times 10}$	$10 \times \underline{10 \times 10 \times 10}$	$10 \times \underline{10 \times 10}.$	$10 \times \underline{10}$	$10 \times \underline{1}$	1
		7	2	0	5	0	4	6

Consider the numeral 7,205,046 which we read as: "seven million, two hundred five thousand, forty-six". (Notice that the word "and" is not used in reading numerals for whole numbers. Otherwise, it would not be clear, for example, when we say "two hundred and five thousand" whether we mean "200 + 5,000" or "205,000".)

We may interpret the numeral 7,205,046 to mean: 7 millions, 2 hundred-thousands, 0 ten-thousands, 5 thousands, 0 hundreds, 4 tens, 6 ones. Since 0 ten-thousands and 0 hundreds both result in zero, these may be omitted in the interpretation. Thus, 7,205,046 means: 7 millions, 2 hundred-thousands, 5 thousands, 4 tens, 6 ones. We also may use an expanded notation form:

7,000,000 + 200,000 + 5,000 + 40 + 6 , or

(7 × 1,000,000) + (2 × 100,000) + (5 × 1,000) + (4 × 10) + (6 × 1) , or

(7 × 10 × 10 × 10 × 10 × 10 × 10) + (2 × 10 × 10 × 10 × 10 × 10) + (5 × 10 × 10 × 10) + (4 × 10) + (6 × 1) .

Problems

5. Write the base ten numeral for each of these expressions.

   a. 7 hundreds, 4 tens, 9 ones.
   b. 8 thousands, 3 hundreds, 6 ones.
   c. 2000 + 700 + 50 + 1
   d. 40,000 + 6000 + 80 + 3
   e. (5 × 1000) + (0 × 100) + (2 × 10) + (4 × 1)
   f. (7 × 10,000) + (6 × 100) + (9 × 1)
   g. (8 × 10 × 10 × 10) + (4 × 10 × 10) + (3 × 1)
   h. (9 × 10 × 10 × 10 × 10) + (5 × 10 × 10 × 10) + (6 × 10)

6. Express each of these base ten numerals in three ways as shown in the illustrative example below.

   Example: 4257 = 4000 + 200 + 50 + 7
   4257 = (4 × 1000) + (2 × 100) + (5 × 10) + (7 × 1)
   4257 = (4 × 10 × 10 × 10) + (2 × 10 × 10) + (5 × 10) + (7 × 1)

   a. 6184   b. 7350   c. 40,702

Grouping by Fours

We are familiar with grouping objects by tens in connection with our decimal place value numeration system. For instance:

	Number of		Base Ten Numeral
	Tens	Ones	
x		1	1
xx		2	2
xxx		3	3
xxxx		4	4
xxxxx		5	5
xxxxxx		6	6
xxxxxxx		7	7
xxxxxxxx		8	8
xxxxxxxxx		9	9
$\boxed{\text{xxxxxxxxxx}}$	1	0	10
$\boxed{\text{xxxxxxxxxx}}$ x	1	1	11
$\boxed{\text{xxxxxxxxxx}}$ xx	1	2	12
$\boxed{\text{xxxxxxxxxx}}$ xxx	1	3	13
$\boxed{\text{xxxxxxxxxx}}$ xxxx	1	4	14
$\boxed{\text{xxxxxxxxxx}}$ xxxxx	1	5	15
$\boxed{\text{xxxxxxxxxx}}$ xxx $\boxed{\text{xxxxxxxxxx}}$	2	3	23

Etc.

Suppose that we agreed to group objects by fours rather than by tens. Suppose, for example, that instead of grouping fourteen objects as

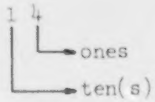
$\boxed{\text{xxxxxxxxxx}}$ xxxx   1 ten and 4 ones

we had grouped the fourteen objects as

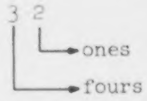
$\boxed{\text{xxxx}}$ $\boxed{\text{xxxx}}$ $\boxed{\text{xxxx}}$ xx   3 fours and 2 ones.

We certainly have not changed the number of objects: fourteen. We have only changed the way in which these fourteen objects are grouped: as "3 fours and 2 ones" rather than as "1 ten and 4 ones".

The numerals of our base ten place value system reflect a tens-and-ones grouping, as



Would it be possible to develop a base four place-value numeration scheme whose numerals reflect a fours-and-ones grouping, as



Let us use sets of one, two, three, . . . , fifteen objects to see how such a base four numeral system might be developed. This is done in the chart below, which includes contrasting base ten numerals.

Note that in the decimal system, each set of ten objects is grouped as 1 ten and the number of these groups is indicated in the tens place. Thus, 23 is 2 tens and 3 ones, and the number of ones left ungrouped is given by the digit 3 . The possible digits in the ones place are then any of the numerals 0, 1, 2, 3, . . . , 9 . Similarly, groups of tens are regrouped into hundreds when there are ten or more of these groups, groups of hundreds are regrouped into thousands when there are ten or more of the hundreds, and so on. Thus, any digit in any place is one of the numerals 0, 1, 2, 3, . . . , 9 . A similar analysis shows that any digit in base four numeration system is one of the numbers 0, 1, 2, 3 since any number of groups exceeding 3 would be regrouped into groups of the next larger size.

	Number of		Base Four Numeral	Base Ten Numeral
	Fours	Ones		
x		1	1	1
xx		2	2	2
xxx		3	3	3
$\boxed{xxxx}$	1	0	10*	4
$\boxed{xxxx}$ x	1	1	11	5
$\boxed{xxxx}$ xx	1	2	12	6
$\boxed{xxxx}$ xxx	1	3	13	7
$\boxed{xxxx}$ $\boxed{xxxx}$	2	0	20	8
$\boxed{xxxx}$ $\boxed{xxxx}$ x	2	1	21	9
$\boxed{xxxx}$ $\boxed{xxxx}$ xx	2	2	22	10
$\boxed{xxxx}$ $\boxed{xxxx}$ xxx	2	3	23	11
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$	3	0	30	12
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$ x	3	1	31	13
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$ xx	3	2	32	14
$\boxed{xxxx}$ $\boxed{xxxx}$ $\boxed{xxxx}$ xxx	3	3	33	15

*This numeral should be read: *one*, *zero*, *base four*. Succeeding numerals in this column would be read: *one*, *one*, *base four*; *one two*, *base four*; *one*, *three*, *base four*; etc.

We now face a problem. What, for instance, does the numeral "13" mean: "1 ten and 3 ones" or "1 four and 3 ones"? We commonly resolve this problem in the following way.

If we see the numeral "13" , for example, we assume that it is written in base ten and understand it to mean "1 ten and 3 ones". This simply follows familiar convention.

If, on the other hand, we wish to write a numeral to convey a base four grouping such as "1 *four* and 3 ones" we agree to use the form

"13_{four}". The subscript "four" indicates the base in which the numeral is written.

On occasion, when showing the base ten numeral for thirteen, for instance, we may write "13_{ten}" instead of simply "13" , just to be certain that there is no misunderstanding. Thus, we agree that

$$13 = 13_{ten} .$$

However, be sure to keep clearly in mind that

$$13 \neq 13_{four}$$

and that

$$13_{ten} \neq 13_{four} .$$

In fact, it is true that

$$13_{ten} = 31_{four}$$

and that

$$13_{four} = 7_{ten} .$$

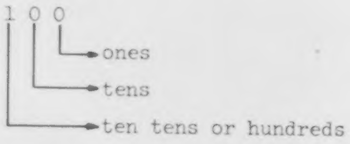
Problems

7. Write "Yes" or "No" to indicate whether each of these is a true statement.
   a. $1_{ten} = 1_{four}$
   b. $2_{four} = 2_{ten}$
   c. $3_{four} = 3$
   d. $10 = 10_{four}$
8. Express each of these base four numerals as base ten numerals.
   a. 21_{four}
   b. 30_{four}
   c. 12_{four}
9. Express each of these base ten numerals as a base four numeral.
   a. 8
   b. 14
   c. 11
10. Using base four numerals:
    a. Name the even whole numbers less than sixteen.
    b. Name the odd whole numbers not greater than fifteen.

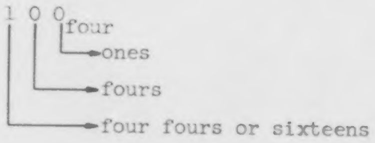
Extending Grouping by Fours

Our base ten numeration system includes more than just two places, a tens place and a ones place. Likewise, a base four numeration system includes more than just a fours place and a ones place. We now consider an extension of grouping by fours.

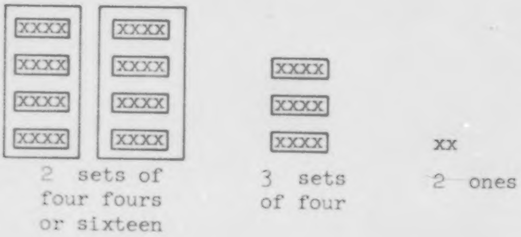
We know that ninety-nine is the greatest whole number that can be named as a two-place numeral in our base ten numeration system: 99 . The next whole number, ten tens, or one hundred, necessitates a new place to the left of tens place. Thus, we name ten tens or one hundred with the numeral


1 0 0
ones
tens
ten tens or hundreds

Similarly, fifteen is the last whole number that can be named with a two-place numeral in a base four numeration system: 33 . The next whole number, four fours, or sixteen, necessitates a new place to the left of fours place. Thus, we name four fours or sixteen with the numeral


100_{four}
ones
fours
four fours or sixteens

The following diagram may help us interpret a numeral such as 232_{four} .


XXXX
XX
2 sets of four fours or sixteen
3 sets of four
2 ones

Thus, 232_{four} is another name for 46_{ten} : $232_{four} = 46$.

The place values associated with a base four numeration system follow the same pattern as do the place values associated with a base ten numeration system, as shown in this chart:

Base × Base × Base	Base × Base	Base	One
Ten × Ten × Ten (Thousands)	Ten × Ten (Hundreds)	Ten	One
Four × Four × Four (Sixty-fours)	Four × Four (Sixteens)	Four	One

Thus, the numeral 2123_{four} may be interpreted as:

$$2123_{four} = (2 \times 4 \times 4 \times 4) + (1 \times 4 \times 4) + (2 \times 4) + (3 \times 1)$$

$$2123_{four} = (2 \times 64) + (1 \times 16) + (2 \times 4) + (3 \times 1)$$

$$2123_{four} = 128 + 16 + 8 + 3$$

$$2123_{four} = 155 \quad (\text{i.e., } 155_{ten}) .$$

Problems

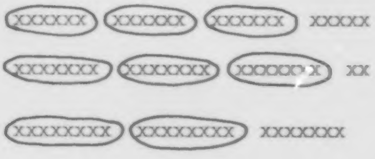
11. Express each of these base four numerals as a base ten numeral.

a. 312_{four} b. 1332_{four} c. 3012_{four}

d. 2301_{four} e. 1230_{four}

Other Bases

A set of objects may be grouped in terms of bases other than ten or four. Consider, for instance, a set of 23 objects that are grouped first by sixes, then by sevens, and then by eights.


xxxxxx xxxxxx xxxxxx xxxxx
xxxxxxx xxxxxxx xxxxxxx xx
xxxxxxxx xxxxxxxx xxxxxxx

3 sixes and 5 ones, or 35_{six}

3 sevens and 2 ones, or 32_{seven}

2 eights and 7 ones, or 27_{eight}

These illustrations point to the fact that the place-value pattern associated with base ten and base four may be applied to other bases as well. For instance:

B × B × B × B	B × B × B	B × B	B*	1
10 × 10 × 10 × 10 10000	10 × 10 × 10 1000	10 × 10 100	10 10	1 1
4 × 4 × 4 × 4 256	4 × 4 × 4 64	4 × 4 16	4 4	1 1
3 × 3 × 3 × 3 81	3 × 3 × 3 27	3 × 3 9	3 3	1 1
2 × 2 × 2 × 2 16	2 × 2 × 2 8	2 × 2 4	2 2	1 1
5 × 5 × 5 × 5 625	5 × 5 × 5 125	5 × 5 25	5 5	1 1
6 × 6 × 6 × 6 1296	6 × 6 × 6 216	6 × 6 36	6 6	1 1
7 × 7 × 7 × 7 2401	7 × 7 × 7 343	7 × 7 49	7 7	1 1
8 × 8 × 8 × 8 4096	8 × 8 × 8 512	8 × 8 64	8 8	1 1
9 × 9 × 9 × 9 6561	9 × 9 × 9 729	9 × 9 81	9 9	1 1

*B denotes base.

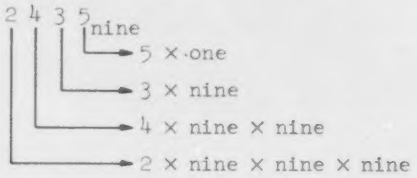
A chart such as the following one may be helpful in showing for the whole numbers one through twenty-five their numerals in each of these bases.

BASE

Ten	Nine	Eight	Seven	Six	Five	Four	Three	Two
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	10
3	3	3	3	3	3	3	10	11
4	4	4	4	4	4	10	11	100
5	5	5	5	5	10	11	12	101
6	6	6	6	10	11	12	20	110
7	7	7	10	11	12	13	21	111
8	8	10	11	12	13	20	22	1000
9	10	11	12	13	14	21	100	1001
10	11	12	13	14	20	22	101	1010
11	12	13	14	15	21	23	102	1011
12	13	14	15	20	22	30	110	1100
13	14	15	16	21	23	31	111	1101
14	15	16	20	22	24	32	112	1110
15	16	17	21	23	30	33	120	1111
16	17	20	22	24	31	100	121	10000
17	18	21	23	25	32	101	122	10001
18	20	22	24	30	33	102	200	10010
19	21	23	25	31	34	103	201	10011
20	22	24	26	32	40	110	202	10100
21	23	25	30	33	41	111	210	10101
22	24	26	31	34	42	112	211	10110
23	25	27	32	35	43	113	212	10111
24	26	30	33	40	44	120	220	11000
25	27	31	34	41	100	121	221	11001

As seen from the chart, the base numeral always appears as 10 when written in that particular base system. Similarly, in a particular base system the numeral 100 always designates the number obtained by multiplying the base by itself.

The place-value pattern for a particular base is used whenever we wish to rewrite a numeral in that base as a base ten numeral. Consider, for instance, the place-value pattern applied to the numeral 2435_{nine}:



In terms of this pattern we may write:

$$\begin{aligned}2435_{\text{nine}} &= (2 \times 9 \times 9 \times 9) + (4 \times 9 \times 9) + (3 \times 9) + (5 \times 1)\\ &= (2 \times 729) + (4 \times 81) + (3 \times 9) + (5 \times 1)\\ &= 1458 + 324 + 27 + 5\\ &= 1814\end{aligned}$$

Suppose that we were concerned with the numeral 2435_{six}, instead of the numeral 2435_{nine}. Then, the base six place-value pattern would permit us to write:

$$\begin{aligned}2435_{\text{six}} &= (2 \times 6 \times 6 \times 6) + (4 \times 6 \times 6) + (3 \times 6) + (5 \times 1)\\ &= (2 \times 216) + (4 \times 36) + (3 \times 6) + (5 \times 1)\\ &= 432 + 144 + 18 + 5\\ &= 599\end{aligned}$$

Problems

12. Express each of these as a base ten numeral.

a. 3421_{five} b. 5674_{eight} c. 4653_{seven}

d. 20122_{three} e. 32012_{four}

A Note about Notation

We have been expressing various nondecimal base numerals as base ten numerals. In this work we moved directly into base ten just as soon as we expressed a nondecimal base numeral in an expanded form. For instance, when we write

$$2134_{\text{five}} = (\underline{2} \times 5 \times 5 \times 5) + (\underline{1} \times 5 \times 5) + (\underline{3} \times 5) + (\underline{4} \times 1)$$

we have expressed all numerals on the right-hand side of the equation in base ten notation.

If for some reason we had wished to express 2134_{five} in an expanded form within base five (rather than in base ten), then we would need to use base five notation throughout the equation. We might convey this idea by writing

$$2134_{\text{five}} = (2 \times 10 \times 10 \times 10)_{\text{five}} + (1 \times 10 \times 10)_{\text{five}} + (3 \times 10)_{\text{five}} + (4 \times 1)_{\text{five}} .$$

These two notations are in keeping with the fact that $5_{\text{ten}} = 10_{\text{five}}$

On still other occasions an expanded form for 2134_{five} might be expressed as

$$2134_{\text{five}} = (2 \times \text{five} \times \text{five} \times \text{five}) + (1 \times \text{five} \times \text{five}) + (3 \times \text{five}) + (4 \times \text{one}) .$$

In such an instance we have expressed the base consistently as the word "five", thus avoiding the place-value numerals 5_{ten} or 10_{five}

In practice we select whichever of these forms is best for a particular purpose.

Summary

The main purpose of this chapter has been to assist in developing a deeper understanding of our Hindu-Arabic numeration system, a decimal or base ten system that utilizes a principle of place value. In addition to a consideration of this system itself, attention was directed to two things that hopefully contributed to this deeper understanding: (1) a modified Greek numeration system which had no place-value principle, and (2) place-value numeration systems having bases other than ten.

This latter material should have clarified the fact that the principles which underlie our Hindu-Arabic numeration system are not determined by the fact that its base is ten. These principles are more general ones which can be applied with other bases as well. The case of the decimal base is but a specific illustration of a more general case.

Throughout this chapter we sought to emphasize that any numeration system is a scheme for naming numbers. Although any particular number may be named in various ways, the properties of a number are not affected by the way in which it is named.

Applications to Teaching

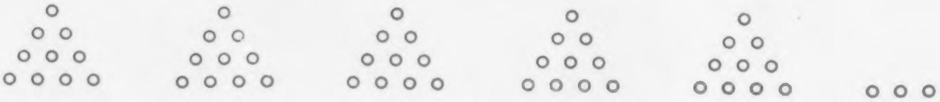
Frequently we display sets of objects in ways that emphasize the decimal base of our numeration system. For instance, we may display a set of 53 objects as 5 rows of 10 objects, and 3 more:

o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
o o o o o o o o o o
o o o

Representations such as this do help children to think about collections of objects in terms of sets of ten "and some more", and consequently direct attention to the decimal base of our numeration system. This is true of any representation that displays collections of objects as sets of ten, regardless of whether they are arranged in rows, in bundles, or whatever.

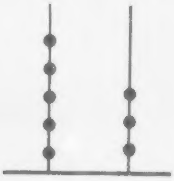
The development of the place value concept is a different and more difficult matter. The place value idea is associated with the numerals we use, and may or may not be reflected in the way in which a set of objects is arranged.

In the numeral 53 the 5 is in tens place and the 3 is in ones place. However, when a set of 53 objects is displayed in rows of ten (and some ones), as above, the display itself does not suggest the idea of a tens place and a ones place in our numeral system. But we may move in the direction of this idea by showing a collection of 53 objects in such a way that sets of 10 are placed at the left of the ones.



With some objects we often show each set of ten as a "bundle" rather than as shown above. In either instance, we show the sets of ten to the left of the ones, "hinting" at the place value idea associated with numerals. We often further this "hinting" by using place value devices in which sets of ten or bundles of ten are placed in "pockets" marked TENS, and remaining single objects are placed in "pockets" marked ONES.

An abacus representation of 53 clearly is associated much more closely with the place value principle.



Here the number of tens and the number of ones are shown by the beads in different positions. The number of tens and the number of ones also may be shown by tally marks (as at the left below) or by digits (as at the right below) in appropriate positions.

Tens	Ones
|||||	|||

Tens	Ones
5	3

We should be aware of the different purposes and uses that are associated with two forms of number charts:

Counting Chart

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Numeral Chart

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49

The Counting Chart highlights ten as the base of our numeration system. If we locate 35 , for instance, on the Counting Chart, it clearly may be associated with 3 rows of 10 "blocks" and 5 "blocks" in the next row.

The Numeral Chart highlights an important feature of the structure of our numerals. The first row of numerals lists the ten basic symbols or digits used in our numeration system. The second row of numerals includes those with 1 in tens place; the third row, those with 2 in tens place, etc.

Each chart has its appropriate place in an instructional program.

If a child is able to complete correctly an example such as

47 = ___ tens + ___ ones

this does not guarantee that he also can complete correctly an example such as

$$4 \text{ tens} + 7 \text{ ones} = \underline{\quad} .$$

The development of an understanding of the place value principle demands that children explore its meaning and application with a variety of representations and in a variety of ways. Suggestions made here regarding numbers less than one hundred can be extended, of course, to apply to numbers greater than ninety-nine.

Exercises - Chapter 8

1. In each ring write $=$ or $>$ or $<$ so that the sentence will be true.
   - a. $7000 + 600 + 50 \bigcirc 7000 + 60 + 5$
   - b. $(3 \times 1000) + (8 \times 100) + (4 \times 1) \bigcirc 3840$
   - c. $1234_{\text{eight}} \bigcirc 1234$
   - d. $4321_{\text{six}} \bigcirc 4321_{\text{five}}$
   - e. $400 \bigcirc 3100_{\text{five}}$
   - f. $3120_{\text{four}} \bigcirc (3 \times 4 \times 4 \times 4) + (1 \times 4 \times 4) + (2 \times 1)$

2. Complete each of the following to make a true sentence.
   - a. $7_{\text{eight}} + 5_{\text{eight}} = \underline{\qquad}_{\text{eight}}$
   - b. $15_{\text{seven}} - 6_{\text{seven}} = \underline{\qquad}_{\text{seven}}$
   - c. $8_{\text{nine}} \times 7_{\text{nine}} = \underline{\qquad}_{\text{nine}}$
   - d. $32_{\text{six}} \div 4_{\text{six}} = \underline{\qquad}_{\text{six}}$

Solutions for Problems

1. a. 47 b. 852 c. 620 d. 403 e. 999

2. a. OC b. YLE c. TJ d. WD e. ZQH

3. No. It is not needed since the system has no place-value principle.

4. a. 2747 b. 81330 c. 460093

5. a. 749 b. 8306 c. 2751 d. 46083
e. 5024 f. 70609 g. 8403 h. 95060

6. a. 6184: 6000 + 100 + 80 + 4
$(6 \times 1000) + (1 \times 100) + (8 \times 10) + (4 \times 1)$
$(6 \times 10 \times 10 \times 10) + (1 \times 10 \times 10) + (8 \times 10) + (4 \times 1)$

b. 7350: 7000 + 300 + 50
$(7 \times 1000) + (3 \times 100) + (5 \times 10)$
$(7 \times 10 \times 10 \times 10) + (3 \times 10 \times 10) + (5 \times 10)$

c. 40702: 40000 + 700 + 2
$(4 \times 10000) + (7 \times 100) + (2 \times 1)$
$(4 \times 10 \times 10 \times 10 \times 10) + (7 \times 10 \times 10) + (2 \times 1)$

7. a. Yes b. Yes c. Yes d. No ($4 = 10_{\text{four}}$)

8. a. 9 b. 12 c. 6

9. a. 20_{four} b. 32_{four} c. 23_{four}

10. a. 0_{four}, 2_{four}, 10_{four}, 12_{four}, 20_{four}, 22_{four}, 30_{four}, 32_{four} .

b. 1_{four}, 3_{four}, 11_{four}, 13_{four}, 21_{four}, 23_{four}, 31_{four}, 33_{four} .

11. a. 54 b. 126 c. 198 d. 177 e. 108

12. a. $(3 \times 125) + (4 \times 25) + (2 \times 5) + (1 \times 1) = 486$

b. $(5 \times 512) + (6 \times 64) + (7 \times 8) + (4 \times 1) = 3004$

c. $(4 \times 343) + (6 \times 49) + (5 \times 7) + (3 \times 1) = 1704$

d. $(2 \times 81) + (1 \times 9) + (2 \times 3) + (2 \times 1) = 179$

e. $(3 \times 256) + (2 \times 64) + (1 \times 4) + (2 \times 1) = 902$

Chapter 9

PREMEASUREMENT CONCEPTS

Introduction

Certain basic geometric figures and concepts have been presented in Chapter 5. Recall that common physical objects provided the foundations on which the development was built. This was so because this is the way in which geometrical ideas are conveyed to young children. Little was said at that time about geometric solids. This topic will now be extended to gain familiarity with associated vocabulary and characteristics as has already been accomplished for many plane figures.

The notion of congruence which has appeared in the earlier discussion will also be a vital concept in the following development. It will provide a means of ordering sets of points which will in turn lead to the concept of measure. By this, we do not mean ordering the points as we have done on the number line. We mean assigning an order to sets of points as for example, among various segments, among plane regions, or solid regions. The corresponding measures are for lengths, areas, and volumes. Thus, we can compare the "sizes" of different geometric objects. The concept of measure will be discussed in Chapter 12. In this chapter, we want first to identify some of the geometrical relationships and configurations by their mathematical names and next, to clarify the concept of ordering sets of points.

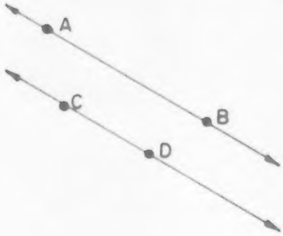
Intersecting and Parallel

The terms intersecting and parallel are familiar through common usage in describing physical phenomena. We speak of a road that runs parallel to a railroad track, or we speak of the intersection of Polk and Oak Streets, and so on. These everyday references describe, although somewhat more loosely, the same relationships that the terms imply in geometry.

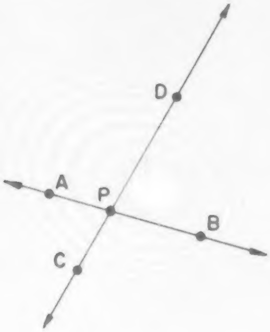
Recall that intersection is one of the set operations dealt with earlier. The intersection of two sets yields a set whose members are those which the two sets have in common. The intersection of two sets, then, can be the empty set or it can have members; it is the empty set if the two are disjoint.

Thinking again of an example of streets, if First Street and Second Street run parallel, there is no intersection. Technically, we would simply say the intersection is empty. However, the less formal description, that "there is no intersection", is often used in geometry for the more accurate description, "the intersection is empty".

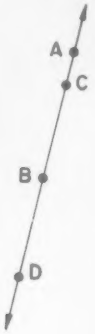
Consider the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ as our two sets of points. The operation of intersection may yield the empty set, a single point, or a line. The drawings illustrate these possible situations.



$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{\}$



$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = P$

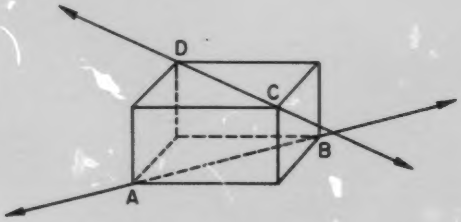


$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \overleftrightarrow{AB}$

In general, "do intersect" or simply "intersects" implies the intersection has members; "do not intersect" implies the intersection is empty.

Although we have only used lines as examples, any sets of points can be considered from the point of view of whether they do or do not intersect. A line may intersect a plane in a line, a point, or not at all; if there is no intersection, the line is said to be parallel to the plane. Two planes may intersect in a line, a plane, or not at all; if they do not intersect, they are said to be parallel.

In space, it is possible that two lines are not parallel and still do not intersect. Picture a bridge over a road as an example. The bridge is not parallel to the road, but does not intersect the road. $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ in this drawing provide another example of nonintersecting, nonparallel lines, $\overleftrightarrow{CD}$ is not parallel to $\overleftrightarrow{AB}$; neither do the two lines intersect.



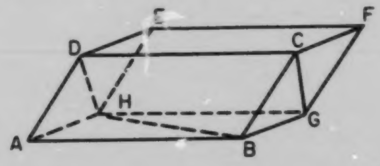
Parallelism for lines may be stated:

> two lines are parallel if they lie in the same plane and do not intersect.

If S and T are sets of points, certain subsets of S and T may be said to be parallel when S and T are parallel. For example, two segments are parallel if they are subsets of parallel lines. Also, two regions are parallel if they are subsets of two parallel planes. A line may be parallel to a plane, and so on. Note that $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$ in the above drawing are subsets of parallel planes but are not considered to be parallel. Lines not lying in the same plane are said to be skew; their intersection is empty. Note also that a plane and a point that is not in the plane may be subsets of parallel planes, but we do not say that the point is parallel to the plane.

Problems*

1. Identify the intersections of the geometrical figures named. They refer to the drawing. If the intersection is the empty set, state whether the figures are parallel or not.



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*Solutions to problems in this chapter are on page 257.

a. $\overline{CD}$ and $\overline{FC}$

b. the plane region with vertices C, D, E, F
and the plane region with vertices A, D,
E, H.

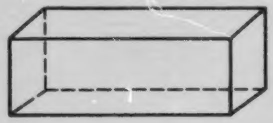
c. $\overline{DH}$ and $\overline{CG}$

d. $\overline{EH}$ and the plane region with vertices
A, B, G, H

e. $\overline{BH}$ and $\overline{EF}$

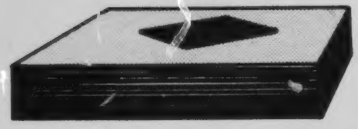
Prisms

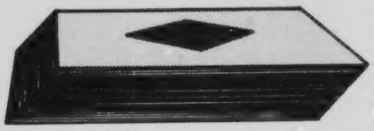
In Chapter 5, a rectangular prism was identified and looked at briefly. It was noted that it was composed of six plane regions called faces. The intersection of any two faces may be empty. If two faces "do intersect", however, their intersection is a segment called an edge. In the same manner, intersecting edges determine a point called a vertex. Thus, the above rectangular prism is the union of its six faces, contains twelve edges and eight vertices. Its shape was abstracted from a rectangular box; all of its faces are rectangular regions.



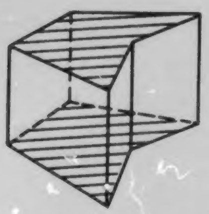
The pictures below of a deck of cards pushed into an oblique position is also a model of a rectangular prism. The criteria for a prism are simply

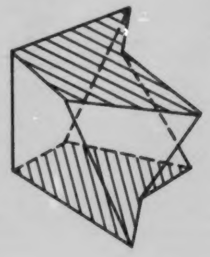
> there are two congruent polygonal regions
> lying in parallel planes, and the edges
> which do not belong to these parallel planes
> are all parallel to one another.

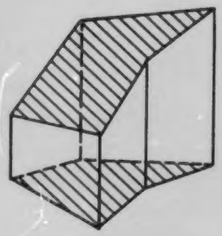




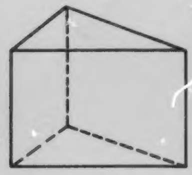
Thus in the figures below, the first is a prism but the other two are not.

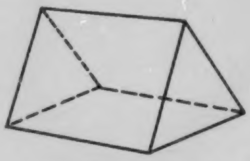

Congruent polygonal regions in parallel planes; edges parallel.


Congruent polygonal regions in parallel planes; edges not parallel.


Edges parallel; polygonal regions not congruent.

The congruent regions in the parallel planes are called *bases* of the prism, and the prism may be identified according to the kind of bases it has. For example, the rectangular prism has rectangular regions for bases; the prism shown in the figure at the left above is a pentagonal prism; either of the figures below is a triangular prism.





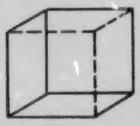
The faces of a prism that are not bases are called the *lateral faces*. Note that each lateral face is a parallelogram region; the boundary of each lateral face consists of two parallel edges called *lateral edges* and two sides of congruent polygons. The two sides of the congruent polygons are also parallel, thus the boundary of each lateral face is a parallelogram.

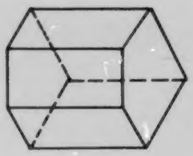
If the bases of a prism are also parallelogram regions, the prism is called a parallelepiped. Thus, the rectangular prisms are a subfamily of the parallelepipeds. A cube, which is the union of six congruent square regions, is another kind of specialized rectangular prism and, hence, is also a parallelepiped. A generic chain of quadrilateral prisms can thus be formed just as was identified for quadrilaterals.

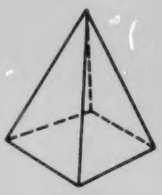
The above two pictures of the deck of cards illustrate another property by which prisms are classified. In the first case, the lateral faces are rectangular regions; in the second drawing they are parallelogram regions only. The first is a right prism; the second is an oblique prism. The lateral faces of right prisms are rectangles. The triangular prisms shown above are right prisms. A cube is a right prism all of whose faces are rectangular regions and, more specifically, are square regions.

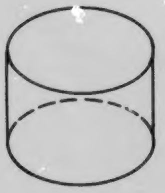
Problems

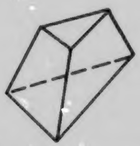
2. a. Select the figures which represent prisms and give the name which best describes each.
   b. For those figures which do not represent prisms, state why they fail to qualify.

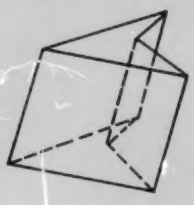

(A)


(B)


(C)

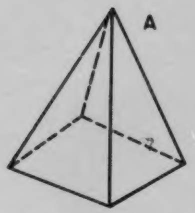

(D)

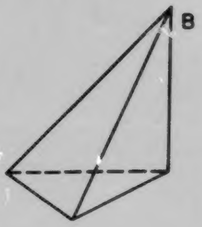

(E)

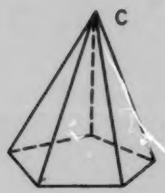

(F)

3. Draw a figure representing an oblique square prism.

Pyramids







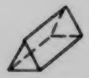
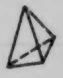
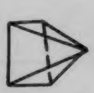
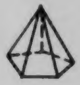
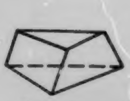
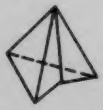
The drawings above represent examples of a familiar set of geometric solids, namely pyramids. As is the case for prisms, there are a great variety of sizes and shapes of pyramids. Each must satisfy these criteria:

> there is a polygonal region called the base;
> there is a point called the apex not in the same plane as the base where all the lateral edges intersect;
> each lateral face is a triangular region determined by the apex and a side of the base.

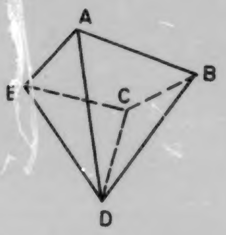
Analogous to the classification of prisms, a pyramid is identified by its base. In the first figure above the base is a square region, and so it is a square pyramid. The others are a triangular pyramid and a pentagonal pyramid respectively. A, B, and C denote their respective apexes.

Problems

4. Which of the following are drawings of pyramids?

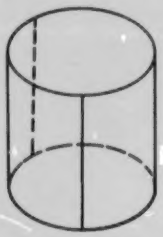
5. a. State an appropriate name for this pyramid.
   b. Identify the apex.
   c. How many edges does it have?
   d. How many faces does it have?

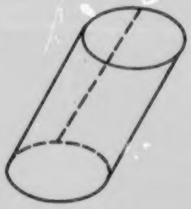


6. What are the possible intersections of two lateral faces of a pyramid?

Cylinders and Cones

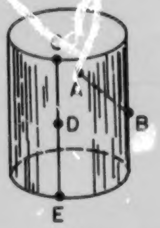
Although we have not discussed all geometric solids that are the union of flat surfaces, we shall now turn our attention to solids with non-flat surfaces. These two figures represent *cylinders*. The two faces





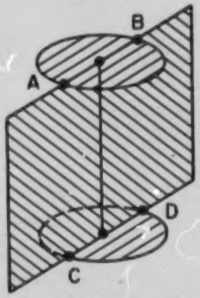
must be congruent regions in parallel planes. They are called *bases* of the cylinder, which is consistent with the other uses of the same term. Although the examples show cylinders with circular bases, this is not a requirement of cylinders in general. At this time we shall not consider cylinders with bases of other configurations, so the discussion will be limited to circular cylinders. The boundaries of the congruent bases are then congruent circles and are *edges* of the cylinder.

The remaining rounded portion of the simple closed surface which defines the cylinder is its *lateral surface*. The distinguishing characteristic of a surface which is not flat is that a segment determined by two of its points is not necessarily a subset of the surface. The drawing below illustrates this feature; $\overline{AB}$ is not a subset of the lateral surface of the cylinder. In fact all points of $\overline{AB}$ between A and B are in the interior of the cylinder.



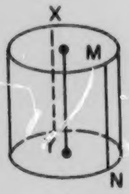
It is possible to find segments which are subsets of the lateral surface of a cylinder, however, such as $\overline{CD}$. In fact, this is a means by which the lateral surface is specified, as we shall show below.

Each of the bases has a center; therefore a segment is determined by these two points. The line containing this segment may be referred to as the line of centers. Consider any plane of which this segment is a subset. It will intersect the bases in two segments called diameters, such as $\overline{AB}$ and $\overline{CD}$ in the figure. Each endpoint of one diameter is to be paired with the appropriate endpoint of the other diameter in order to be able to describe the set of points in the lateral surface. The "appropriate" endpoints of the respective diameters are those which determine a segment that does not intersect the line of centers. Thus, in the drawing, A is paired with C and B is paired with D .

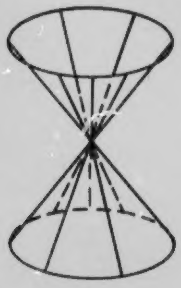


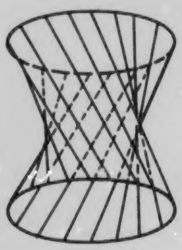
By considering a different plane, we will obtain two new pairs of points. If all such planes are conceived, all such pairs are generated. Then we say we have defined a correspondence between the points in the boundaries of the two congruent bases. Any two points which are thus paired are corresponding points.

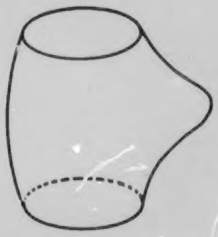
Each of these pairs of corresponding points determines a segment parallel to the segment connecting the centers. The union of all segments determined by corresponding points is the set of points in the desired surface. Each segment is said to be an element of the cylinder. Any two elements are parallel. In the figure, $\overline{MN}$ and $\overline{XY}$ are elements and therefore are parallel.



The preceding description for generating the lateral surface is rather involved. This is because we want to specify the particular correspondence we have in mind since other possible configurations can be formed with the required bases. If a different correspondence were defined between the points of the boundaries, a figure as in (a) and (b) below might evolve. If no segments were specified, the resulting figure might be as in (c).


(a)


(b)


(c)

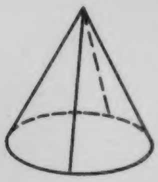
We can now state that a circular cylinder must satisfy these criteria:

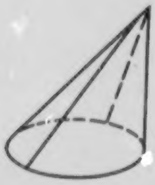
> there are two congruent circular regions in parallel planes;
> there is a surface which is the union of all segments determined by corresponding points of the boundaries of the bases.

Referring back to our first two examples of cylinders in this section, the first is a right circular cylinder; the second is oblique. In order to be right, any element of the cylinder must form right angles with each segment of a base which intersects it.

It is apparent on reflection that there is a distinct similarity between the cylinder and the prism. They each have congruent regions in parallel planes for bases. If an appropriate correspondence were set up between the points of the sides of the bases of a prism, and if line segments joining them were considered such that they are parallel, then the lateral faces would be specified. In fact, the only difference is that the bases of a prism must be polygonal regions while those of a circular cylinder must be circular regions. It is the case that a cylinder can be defined in such a way as to include prisms as a subfamily of cylinders; however, this will not be done for the elementary level.

By the same token that cylinders are analogous to prisms, *cones* are analogous to pyramids. As with cylinders, we will restrict the plane region of a cone to a circular shape and designate it as the *base* of the cone. The point which is not in the same plane as the base describes the *apex*. The *lateral surface* is not so difficult to describe in this figure. It is simply the set of line segments determined by the apex and each point of the circular boundary of the base.





Problems

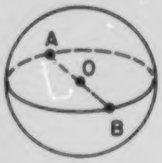
7. State a definition of cylinders so that prisms would be a subfamily of cylinders, namely polygonal cylinders.
8. Describe or draw representations of the intersections of a plane and a right circular cylinder if the plane does intersect the cylinder and is
    a. parallel to the bases;
    b. parallel to the line of centers;
    c. not parallel to the base nor the line of centers

Spheres

The final solid to be included is the *sphere*. As is the case for a circle, a sphere has a *center*. All segments connecting the center of the sphere and a point on the sphere are congruent. Indeed, this specifies the set of points in the sphere. They are:

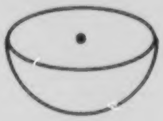
> all endpoints of congruent segments
> which have one endpoint in common,
> but not including their common endpoint.

The congruent segments are radii (singular: radius). The union of two radii which are each subsets of the same line is a diameter. In the figure, O is the center, $\overline{AO}$ and $\overline{OB}$ are radii and therefore congruent,



and $\overline{AB}$ is a diameter.

A hemisphere is half of a sphere. Any plane that contains the center of a sphere will "cut off" a hemisphere.



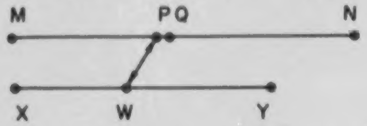
Problems

9. Identify the intersection of
   a. a plane and a sphere;
   b. the center and the sphere;
   c. a diameter and the sphere;
   d. the center of the sphere and one of its hemispheres.

Ordering Sets of Points

The ordering of sets is not a new topic. Chapter 2 was devoted to the comparison of sets according to order and certain properties of ordered sets. The approach taken was to pair the members of the two sets in question. Then it was possible to decide whether one set had more or fewer members than the other or whether the two sets were equivalent. If we try to use the same process with sets of points, two difficulties are encountered which make the procedure impossible.

Take, for example, two segments, $\overline{MN}$ and $\overline{XY}$. Each is an infinite set, and therefore if we began pairing points we would never exhaust the points of either set. This alone eliminates pairing as a means of ordering. To compound the problem, however, there is another property of segments that defies pairing; that is that they are continuous. If two points, say P and W, are chosen and paired, we cannot select the point next to P to pair with the point next to W, because there are no next points on a segment. If for instance, Q is named in $\overline{MN}$ there are an infinite number of points between P and Q, so nothing has been accomplished.

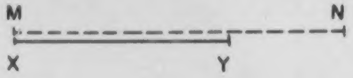


Then, how are segments, and sets of points in general, ordered? We can resort to our concept of congruence to assist us. It has been established intuitively that two line segments are congruent if a movable copy of one can be matched and fitted exactly on the other. A similar procedure serves to indicate whether curves, polygons, plane regions and so on are congruent. It does not prove useful in determining whether or not solid figures are congruent, however, since a movable copy of a solid cannot always be matched and fitted exactly on the other intact. For example, a solid block cannot be fitted into another solid block.

If two sets of points are not congruent, we can still conceive of an order between them. Suppose you measure the dimensions of this book. Its length is shorter than one meter. You are essentially carrying out a comparison of set size with the aid of a movable copy. The sets being compared are an edge of your book and a platinum bar in the United States Bureau of Standards in Washington, D. C. The movable copy is a meter stick and its scale is a record of the length of the bar. By stating that the length of the edge of the book is shorter than one meter, we are ordering the sizes of two physical representations of line segments. In particular, your book is shorter than the bar.

Geometrical segments are handled in a similar fashion. Suppose it is desired to order the two sets $\overline{MN}$ and $\overline{XY}$. We make a copy of $\overline{MN}$, indicated by the dotted segment, and lay it over $\overline{XY}$. We have already

said that if they fit exactly, $\overline{MN}$ and $\overline{XY}$ would be congruent. If, however, they do not, one of two situations must exist. $\overline{XY}$ will be congruent to a proper subset of $\overline{MN}$ or $\overline{MN}$ will be congruent to a proper subset of $\overline{XY}$. In the first instance, we would say $\overline{XY}$ *is shorter than* $\overline{MN}$ or, equivalently, $\overline{MN}$ *is longer than* $\overline{XY}$. The second possibility is interpreted as $\overline{MN}$ is shorter than $\overline{XY}$ or $\overline{XY}$ is longer than $\overline{MN}$. Our example demonstrates the first case, since $\overline{XY}$ is congruent to a proper subset of $\overline{MN}$. We can order the sets by



$\overline{XY}$, $\overline{MN}$ in increasing order.

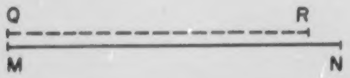
For finite sets, A and B , recall that comparing sets assured exactly one of three possible outcomes:

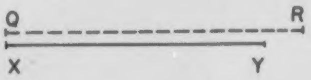
A is equivalent to B ;
A has more members that B ;
A has fewer members than B .

Now we can state the parallel relationships for infinite sets of points, $\overline{AB}$ and $\overline{CD}$:

$\overline{AB}$ is congruent to $\overline{CD}$;
$\overline{AB}$ is longer than $\overline{CD}$;
$\overline{AB}$ is shorter than $\overline{CD}$.

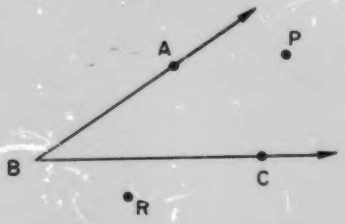
Note that "$\overline{AB}$ is longer than $\overline{CD}$" does *not* mean $\overline{AB}$ has more members than $\overline{CD}$. We are saying nothing about "how many" in relating infinite sets. By repeated comparison, it is possible to order more than two segments. Thus QR below would fit into the order $\overline{XY}$, $\overline{QR}$, $\overline{MN}$ as the diagram illustrates. We find that $\overline{QR}$ is congruent to a subset of $\overline{MN}$, and that $\overline{XY}$ is congruent to a subset of $\overline{QR}$. Therefore $\overline{QR}$ is shorter than $\overline{MN}$ and $\overline{QR}$ is longer than $\overline{XY}$.

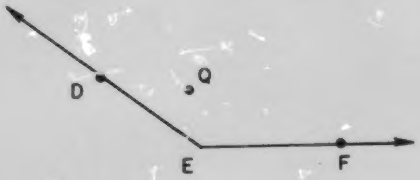




In Chapter 12, these order relationships will be restated in terms of numbers associated with segments. These numbers will be the measures of the segments. By our ordering, however, we have done no measuring.

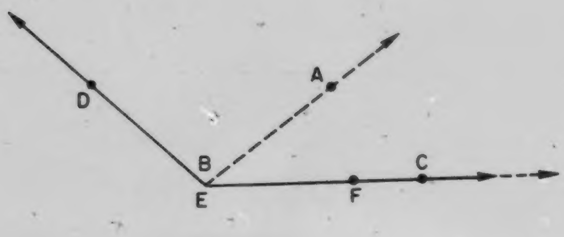
The second kind of geometric figures that we wish to order is angles. An angle is the set of points defined by the union of two rays, not subsets of the same line, which have a common endpoint. Just as simple closed curves separate a plane into three subsets (the curve, its interior and its exterior), angles can be thought of as doing the same thing. A point is in the interior of an angle if it lies between two points, one on each ray, exclusive of the vertex. Thus P is in the interior of ∠ABC and Q is in the interior of ∠DEF . P is in the exterior of ∠DEF and R is an exterior point of ∠ABC .





To order two angles, we rely on a movable copy of one in much the same manner as we did for segments. For the angles pictured above, we could place a copy of ∠ABC over ∠DEF so that one side of the copy coincides with one side of ∠DEF . The figure below shows one way the copy can be positioned. If the second side of the copy also coincides with the second side of ∠DEF , we would say

∠ABC is congruent to ∠DEF .



If it is not possible to get such a coincidence, as it is not for our angles, we define an order. Note that the points of $\overrightarrow{BA}$, except the endpoint B, lies in the interior of $\angle DEF$. Whenever this phenomenon holds, we say

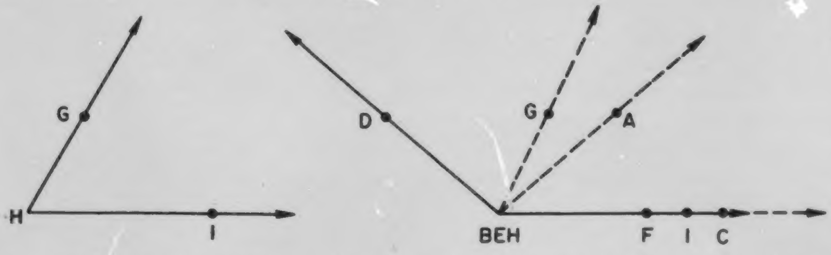
$$\angle ABC \text{ \underline{is} \underline{smaller} \underline{than} } \angle DEF$$

or, equivalently, $\angle DEF$ is larger than $\angle ABC$. If it happened that the interiors of the two angles have points in common and that $\overrightarrow{BA}$, except for B, were a subset of the exterior of $\angle DEF$, then

$$\angle ABC \text{ is larger than } \angle DEF$$

or $\angle DEF$ is smaller than $\angle ABC$.

Considering a third angle, $\angle GHI$, we find that $\overrightarrow{GH}$, except for G, lies in the exterior of $\angle ABC$ and in the interior of $\angle DEF$. Two statements expressing this are $\angle GHI$ is larger than $\angle ABC$, and $\angle GHI$ is smaller than $\angle DEF$. In increasing order, we could write $\angle ABC$, $\angle GHI$, $\angle DEF$. As for segments, this procedure can be repeated indefinitely for as many angles as we wish. Congruent angles would occupy the same position in the order.

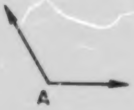


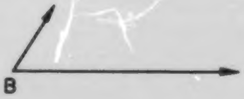
The definition of measurement for angles will not be included in Chapter 12 because it is not treated in the K-1 text materials. It has been discussed here to indicate that the ordering of sets of points can be accomplished for figures other than segments. It is actually possible to use congruence as a means of ordering regions and solids also, although it is a bit more complicated. It is not possible, however, to order unlike sets of points; that is, we cannot order segments and angles, nor segments and plane regions, and so on.

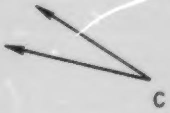
Problems

10. Represent $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$ such that their order from shortest to longest is $\overline{CD}$, $\overline{AB}$, $\overline{EF}$.

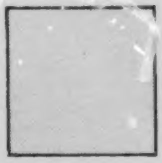
11. Place the sets represented by the angles below in increasing order.

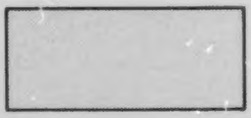






12. Can you devise a means of ordering the two regions shown below?





Applications to Teaching

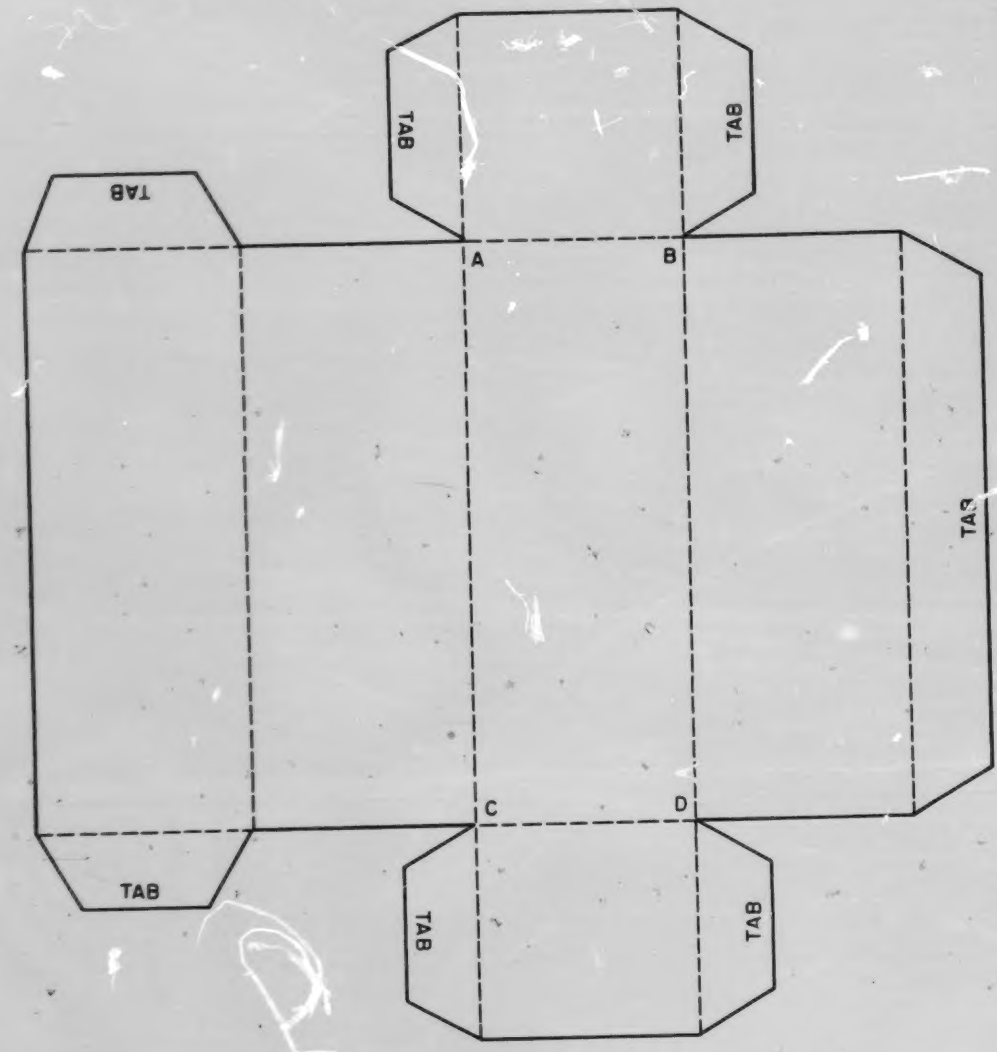
Teachers have found it most helpful to have in the room a wide collection of objects which illustrate geometrical solids. Children also enjoy bringing such objects from home. Effective ways of using these and other models have been recommended in this section of Chapter 5.

On the next pages are included four patterns to be used in constructing geometrical solids out of paper. Having the children observe your demonstration of a construction emphasizes two aspects of solids. Many are the union of plane regions that do not lie in the same plane, and they are hollow.

The ideas in the pre-measurement section are most important. The children should be asked to participate as much as possible in manipulating figures to compare their sizes, both to understand congruence and order. They often experience some difficulty in visualizing congruent regions if they have different orientations, so practice should be provided with this in mind.

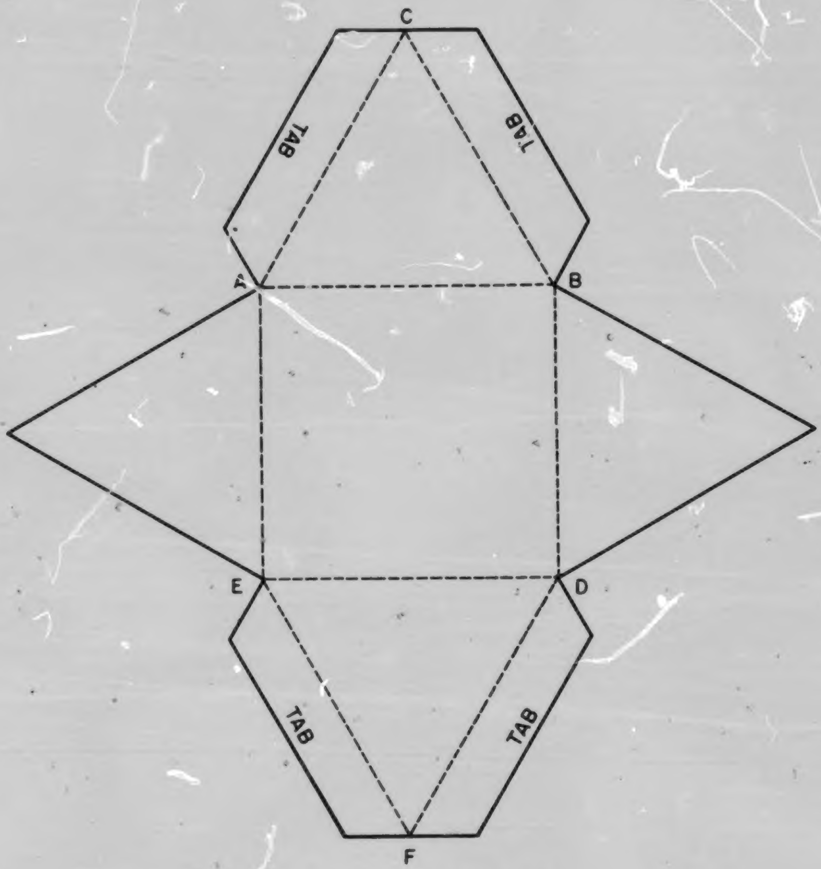
PRISM - Construction of a square prism.

1. Draw a rectangle with vertices A , B , C , D as shown.
2. Draw, as shown, three other rectangles congruent to the rectangle already drawn with tabs.
3. Draw the two squares along $\overline{AB}$ and $\overline{DC}$ with tabs, as shown.
4. Cut around the boundary of the figure and fold along the dashed line segments.
5. Use scotch tape or paste to hold the model together. The tabs will help give rigidity to the model. You may want to trim them some if you use scotch tape.
6. The bases of this rectangular prism are squares, hence the name - square prism.
7. This picture has been reduced photographically. The original had the length of $\overline{AB}$ as 1 1/2" and that of $\overline{BC}$ as 4" . This made a 1 1/2" × 1 1/2" × 4" square prism.



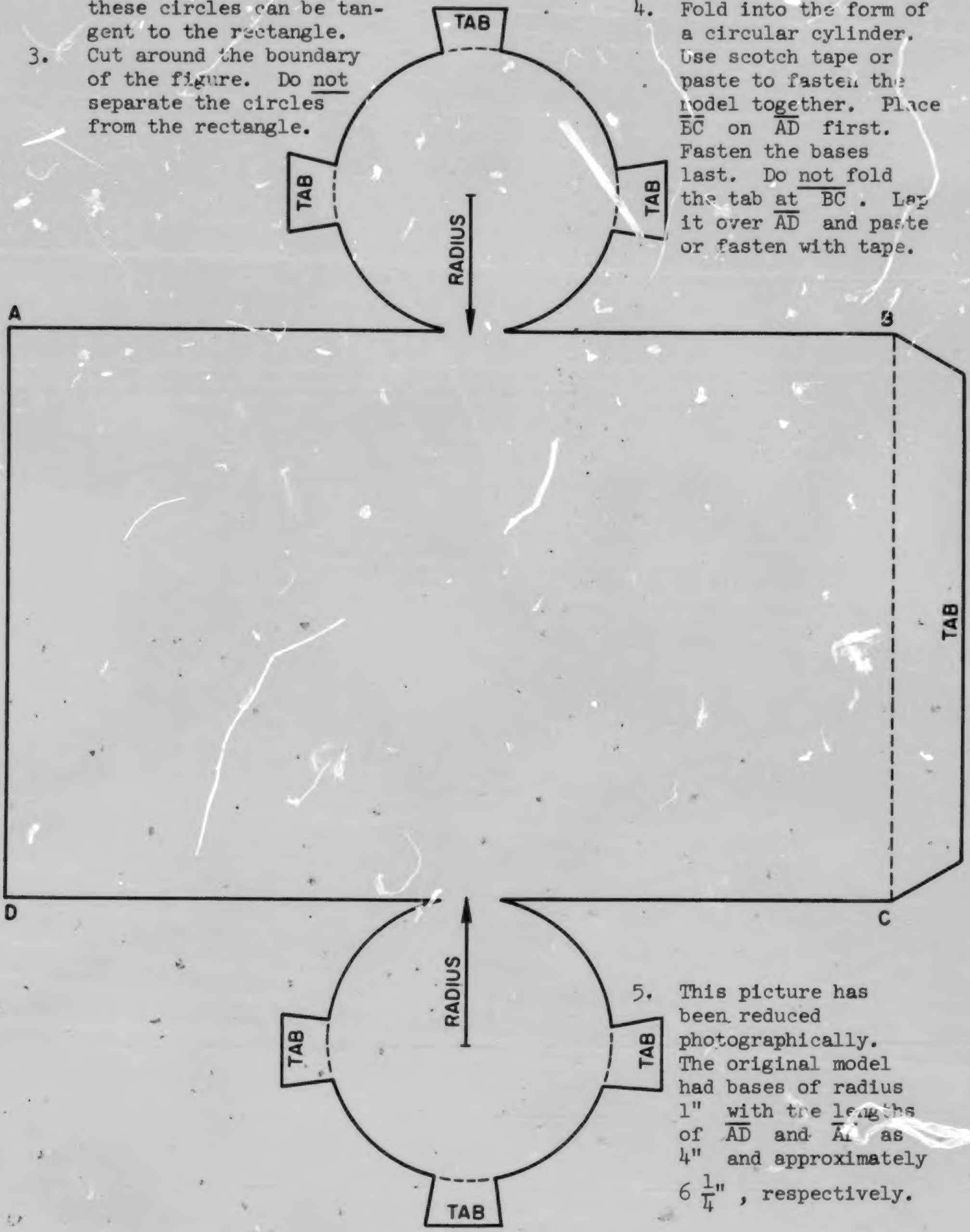
PYRAMID - Construction of a square pyramid.

1. Draw a square with vertices A , B , D , E as shown.
2. Draw the arcs with centers at A and B and radius $\overline{AB}$. Label the intersection shown as C .
3. Draw dashed line segments $\overline{AC}$ and $\overline{BC}$ to form "dashed" equilateral triangle with vertices A , B , C . Draw tabs as shown.
4. Repeat step 3 to obtain "dashed" equilateral triangle with vertices E , D , F with tabs as shown.
5. Draw the equilateral triangle shown on $\overline{BD}$ and $\overline{AE}$.
6. Cut around the boundary and fold along the dashed line segments.
7. Fasten with scotch tape or paste. The tabs will help in putting the model together. You may want to trim some of them if you use scotch tape.
8. This picture has been reduced photographically. The original model had the lengths of $\overline{AB}$ as 2" .



CYLINDER - Construction of a circular cylinder.

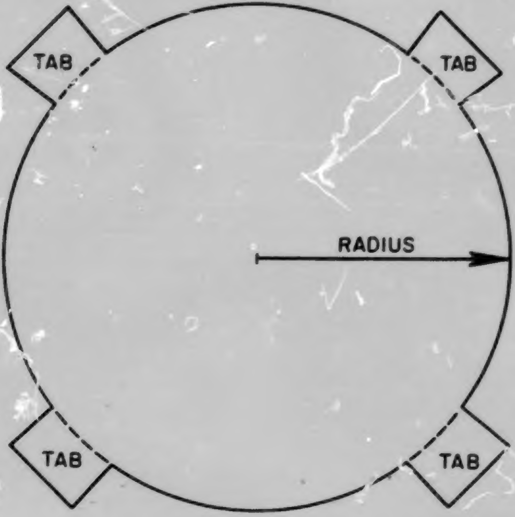
1. Draw the rectangle with vertices A , B , C , D .
2. Draw two congruent circles with radius as shown. In order to make the model easier to construct, these circles can be tangent to the rectangle.
3. Cut around the boundary of the figure. Do *not* separate the circles from the rectangle.
4. Fold into the form of a circular cylinder. Use scotch tape or paste to fasten the model together. Place $\overline{BC}$ on $\overline{AD}$ first. Fasten the bases last. Do *not* fold the tab at $\overline{BC}$. Lap it over $\overline{AD}$ and paste or fasten with tape.

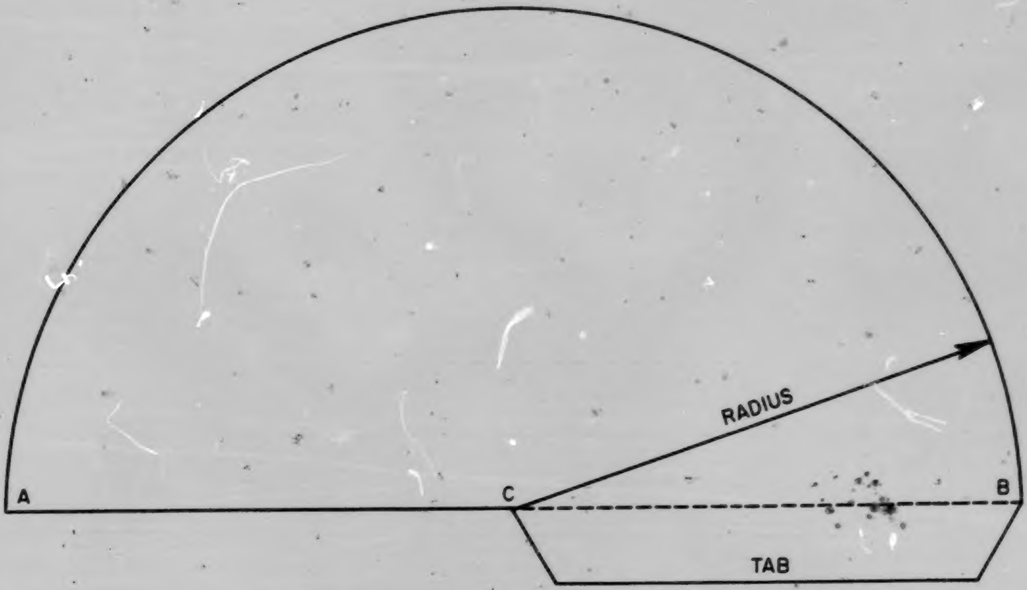


5. This picture has been reduced photographically. The original model had bases of radius 1" with the lengths of $\overline{AD}$ and $\overline{AB}$ as 4" and approximately $6\frac{1}{4}$" , respectively.

CONE - Construction of a circular cone.

1. Use a compass to draw a circle with a radius as shown in the diagram. Draw tabs as shown.
2. Cut around the boundary of this figure. The circular region will be the base of the cone.
3. Use a compass to draw a semicircle with a radius as shown in the diagram. C is the center of the circle. $\overline{AB}$ is a diameter. Draw the tab as shown.
4. Cut around this figure.
5. Fasten $\overline{AC}$ to $\overline{BC}$ with scotch tape or paste so that $\overline{AC}$ falls on $\overline{BC}$.
6. Fasten the base to this model by folding the tabs and using scotch tape or paste.





Exercises - Chapter 9

1. Why is the following definition of parallel segments not sufficient to determine what we mean by parallel segments?

   > Two segments are parallel if they lie
   > in the same plane and do not intersect.

2. What are the sets which may result in the intersection of a line and a plane?

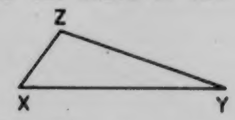
3. Construct a paper model of a square pyramid using the pattern on page 253.

4. a. How many edges does a triangular pyramid have?
   b. How many edges does a rectangular pyramid have?
   c. If the base of a pyramid has n sides, how many edges does the pyramid have?

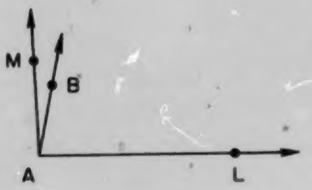
5. Identify by a drawing the intersection of a plane parallel to $\overline{AO}$ and the cone, if A is the apex and O is the center of the base. Assume the plane intersects the cone in more than one point.

6. Which of the following solid regions must be convex sets?
   a. sphere; b. circular cylinder; c. quadrilateral pyramid.

7. State in increasing order the sides of the triangle.



8. Why is it incorrect to say $\overrightarrow{AB}$ is a subset of the interior of $\angle MAL$?



Solutions for Problems

1. a. C b. $\overline{DE}$ c. { } ; they are parallel d. H e. { } ; not parallel.

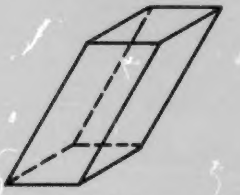
2. a. (A) cube; (B) right pentagonal prism; (F) non-convex quadrilateral prism.

   b. (C) There are not 2 congruent, parallel bases; the lateral edges are not parallel.

      (D) The congruent faces are not polygonal; the lateral surface is not the union of parallelogram regions.

      (E) The parallel bases are not congruent; the lateral edges are not parallel.

3.



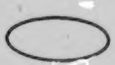
4. (b) , (c) , (d) , (f)

5. a. quadrilateral pyramid
   b. D
   c. 8
   d. 5

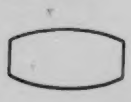
6. A lateral edge or the apex

7. A cylinder is a geometric solid which is the union of two si[illegible]rly oriented parallel regions whose boundaries are simple closed curves and all the segments determined by corresponding points of the congruent boundaries.

8. a. a circle; b. a rectangle or a segment congruent to the segment connecting the centers; c.

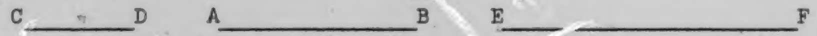


or



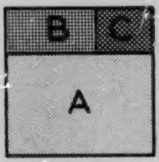
9. a. a circle, a point, or { } ; b. { } ; the center is not part of the sphere; c. two points--the endpoints of the diameter; d. { } .

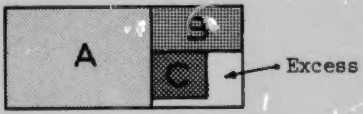
10.



11. $\angle C$, $\angle B$, $\angle A$

12. We can partition one region, make movable copies and lay them on the other region. If they fit, we will say they have the same size. If they do not, one will be larger than the other.





Thus, the rectangular region is larger than the square region.

Chapter 10

ADDITION AND SUBTRACTION TECHNIQUES

Introduction

We have used sets to describe addition and subtraction and to develop its properties. Knowing that $5 + 3$ is the number of members in $A \cup B$, where A is a set of 5 members and B is a disjoint set of 3 members, enables us to count the members of $A \cup B$ and to discover that $5 + 3$ is 8. Knowing that $5 + 3 = 8$, from the definition of subtraction, we can see that $8 - 3 = 5$. This is fine, but it does not really help us much if we want to determine $892 + 367$ or $532 - 278$. To do problems like these quickly and accurately is a goal of real importance. It is a goal whose achievement is made much easier in our decimal system of numeration than in, for instance, the Chinese or Egyptian systems.

This chapter is concerned with explaining the whys and wherefores of so-called "carrying" and "borrowing" in the processes of computing sums and differences. Regrouping is a more accurate term for "carrying" and "borrowing" and will be used throughout this text.

We must recall how our system of numeration with base ten is built. What does the numeral 532 stand for? It stands for $500 + 30 + 2$; or 5 hundreds + 3 tens + 2 ones; or again, since one hundred stands for 10 tens, 532 stands for 5 groups of ten tens + 3 groups of ten + 2 ones. Also if we know that a number has 2 groups of ten tens and 7 groups of ten and 8 ones, we can write a numeral for that number in the form $(2 \times [10 \times 10]) + (7 \times 10) + (8 \times 1)$ or $200 + 70 + 8 = 278$. When we write the numeral in this stretched-out way, we have written it in expanded form.

Regrouping Used in Addition

Let us assume that we know the addition facts for all the one-digit whole numbers and that we understand our decimal system of numeration. How does this help us? Let's try some examples. Suppose we want the sum of 42 and 37. Since we are adding (4 tens + 2 ones) and

(3 tens + 7 ones) we get (7 tens + 9 ones) which we can write as 79 .

Essentially what we are doing is finding how many groups of tens and how many units we have and then using our system of numeration to write the correct numeral. We may show this in several different forms or algorithms, such as:

(a)
$$\boxed{\begin{array}{l} 3 \text{ tens} + 7 \text{ ones} \\ \underline{4 \text{ tens} + 2 \text{ ones}} \\ 7 \text{ tens} + 9 \text{ ones} = 79 \end{array}}$$

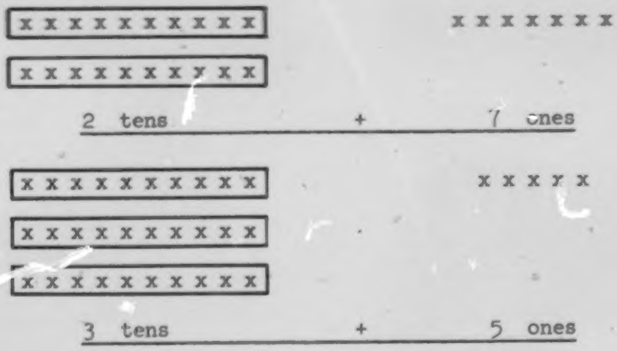
(b)
$$\boxed{\begin{array}{l} 30 + 7 \\ \underline{40 + 2} \\ 70 + 9 = 79 \end{array}}$$

(c)
$$\boxed{\begin{array}{rl} 37 & \\ +\underline{42} & \\ 9 & (7 + 2) \\ \underline{70} & (30 + 40) \\ 79 & \end{array}}$$

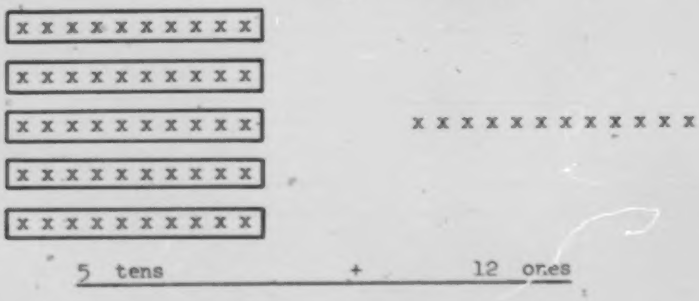
Or we may use an equation form such as

$$\begin{aligned} 37 + 42 &= (30 + 7) + (40 + 2) \\ &= (30 + 40) + (7 + 2) && \text{Applying the associative} \\ &= 70 + 9 && \text{and commutative properties} \\ &= 79 \end{aligned}$$

Let us now add 27 and 35 . This time we have (2 tens + 7 ones) + (3 tens + 5 ones) which may be illustrated:


x x x x x x x x x x
x x x x x x x x x x
x x x x x x x
2 tens + 7 ones
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x
3 tens + 5 ones

By putting these groups together we now have:


x x x x x x x x x x
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x x x x x x x x x x
x x x x x x x x x x
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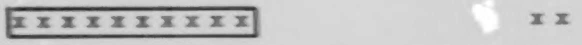
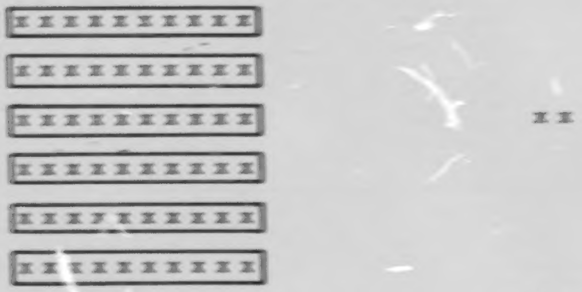
5 tens + 12 ones

We now regroup the 12 ones and get another set of 1 ten and 2 ones.



1 ten + 2 ones

We now add (5 tens + 1 ten) + 2 ones.



5 tens + 1 ten + 2 ones
= 6 tens + 2 ones = 62

Or, algorithms such as these may be used:

(a)
2 tens + 7 ones
3 tens + 5 ones
5 tens + 12 ones, or
5 tens + 1 ten + 2 ones, or
6 tens + 2 ones = 62

(b)
20 + 7
30 + 5
50 + 12, or
50 + 10 + 2, or
60 + 2 = 62

(c)
27
+ 35
12 (7 + 5)
50 (20 + 30)
62

Using an equation form we may write:

$$\begin{aligned}27 + 35 &= (20 + 7) + (30 + 5)\\ &= (20 + 30) + (7 + 5) \quad \text{Applying the associative and commutative properties}\\ &= 50 + 12\\ &= 50 + (10 + 2)\\ &= (50 + 10) + 2 \quad \text{Applying the associative property}\\ &= 60 + 2\\ &= 62\end{aligned}$$

We may extend these same ideas to the addition of two whole numbers, each greater than 100 . Suppose, for instance, that we were adding 568 and 275 :

(a)
$$\begin{array}{|l|}
\hline
5 \text{ hundreds} + 6 \text{ tens} + 8 \text{ ones} \\
\underline{2 \text{ hundreds} + 7 \text{ tens} + 5 \text{ ones}} \\
7 \text{ hundreds} + 13 \text{ tens} + 13 \text{ ones, or} \\
7 \text{ hundreds} + 14 \text{ tens} + 3 \text{ ones, or} \\
8 \text{ hundreds} + 4 \text{ tens} + 3 \text{ ones} = 843 \\
\hline
\end{array}$$

or we may write

(b)
$$\begin{array}{|l|}
\hline
500 + 60 + 8 \\
\underline{200 + 70 + 5} \\
700 + 130 + 13 \text{ , or} \\
700 + 140 + 3 \text{ , or} \\
800 + 40 + 3 = 843 \\
\hline
\end{array}$$

or (c)
$$\begin{array}{|rl|}
\hline
568 & \\
+\,\underline{275} & \\
13 & (8 + 5) \\
130 & (60 + 70) \\
\underline{700} & (500 + 200) \\
843 & \\
\hline
\end{array}$$

Precisely the same process is used in adding three or more numbers. Once again the properties of addition are important. Thus:

563 + 787 + 1384 can be thought of as follows:

$$\begin{array}{rcrcr}
563 & = & 500 + 60 + 3 & = & (5 \times 100) + (6 \times 10) + (3 \times 1) \\
787 & = & 700 + 80 + 7 & = & (7 \times 100) + (8 \times 10) + (7 \times 1) \\
\underline{1384} & = & \underline{1000 + 300 + 80 + 4} & = & \underline{(1 \times 1000) + (3 \times 100) + (8 \times 10) + (4 \times 1)} \\
 & & & & (1 \times 1000) + (15 \times 100) + (22 \times 10) + (14 \times 1)
\end{array}$$

and the sum 563 + 787 + 1384

$$\begin{aligned}
&= (1 \times 1000) + (15 \times 100) + (22 \times 10) + (14 \times 1) \\
&= (1 \times 1000) + [(1 \times 1000) + (5 \times 100)] + [(2 \times 100) + \\
&\qquad (2 \times 10)] + [(1 \times 10) + (4 \times 1)] \\
&= [(1 \times 1000) + (1 \times 1000)] + [(5 \times 100) + (2 \times 100)] + \\
&\qquad [(2 \times 10) + (1 \times 10)] + (4 \times 1) \\
&= (2 \times 1000) + (7 \times 100) + (3 \times 10) + (4 \times 1) \\
&= 2000 + 700 + 30 + 4 \\
&= 2734
\end{aligned}$$

This is usually abbreviated a great deal. But it is important that the underlying pattern be understood and the abbreviations recognized. Thus:

$$\begin{array}{r} 500 + 60 + 3 \\ 700 + 80 + 7 \\ \underline{1000 + 300 + 80 + 4} \\ 1000 + 1500 + 220 + 14 \end{array}$$

can be written with partial sums indicated as:

$$\begin{array}{rl} 563 & \\ 787 & \\ \underline{1384} & \\ 14 & \text{sum of ones} \\ 220 & \text{sum of tens} \\ 1500 & \text{sum of hundreds} \\ \underline{1000} & \text{sum of thousands} \\ 2734 & \end{array}$$

and the operation may be still further abbreviated to:

$$\begin{array}{r} \textcircled{1}\textcircled{2}\textcircled{1} \\ 563 \\ 787 \\ \underline{1384} \\ 2734 \end{array}$$

Finally, by omitting even the "carry over" numerals we get:

$$\begin{array}{r} 563 \\ 787 \\ \underline{1384} \\ 2734 \end{array}$$

Problems*

1. Find the sum, 38 + 73 + 22 , by an algorithm that shows clearly how the sum is obtained from the addition facts for 0 through 9 only.

2. Show the individual steps required in applying the associative and commutative laws to show that

$$(30 + 7) + (50 + 8) = (30 + 50) + (7 + 8) .$$

A Property of Subtraction

Just as we worked the same problem by various methods to get an insight into the addition process, we shall now study the subtraction process by examining various techniques. Let us use a simple example to illustrate the procedures.

Using an equation form for finding the value of the unknown addend n in $n + 23 = 58$ and comparing this with the usual algorithm identifies a property of subtraction that is used extensively in computational work. We write:

$$58 - 23 = (50 + 8) - (20 + 3) .$$

The property of subtraction that deserves our special attention is that which will enable us to express $(50 + 8) - (20 + 3)$ in a useful form.

---

*Solutions for problems in this chapter are on page 271 .

The usual procedure for subtracting is by the vertical alignment,

$$\begin{array}{r}58\\ \underline{23}\end{array},$$

which may be expressed as either of the following:

(a) $$\boxed{\begin{array}{l}5 \text{ tens} + 8 \text{ ones}\\ \underline{2 \text{ tens} + 3 \text{ ones}}\\ 3 \text{ tens} + 5 \text{ ones} = 35\end{array}}$$

(b) $$\boxed{\begin{array}{l}50 + 8\\ \underline{20 + 3}\\ 30 + 5 = 35\end{array}}$$

In the algorithm (b) above, notice that 3 is subtracted from 8 and 20 is subtracted from 50 to arrive at the tens and ones in the difference. In equation form, this entire process is written:

$$\begin{aligned}58 - 23 = (50 + 8) - (20 + 3) &= (50 - 20) + (8 - 3)\\ &= 30 + 5\\ &= 35\end{aligned}$$

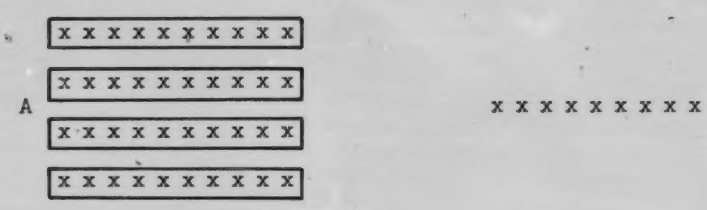
We may state the property, which allows $(50 + 8) - (20 + 3)$ to be reexpressed as $(50 - 20) + (8 - 3)$, more generally in the following way:

> If $a + b$ is the name of one number and
> $c + d$ is the name of a second number,
> and if $a \geq c$ and $b \geq d$, then
> $(a + b) - (c + d) = (a - c) + (b - d)$

We shall see repeated use of this property, along with regrouping, throughout the rest of this chapter.

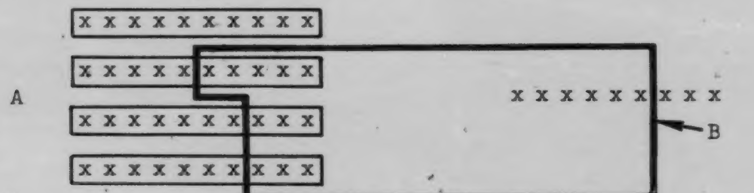
Next, let us interpret subtraction, such as 17 from 49, in terms of set removal. From a set, A, of 49 objects remove a subset, B, of 17 objects, leaving a remainder set, $A \sim B$, whose number is to be specified.

We can take for A a collection of 49 x's arranged as follows:



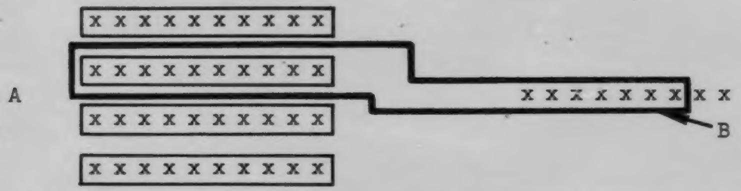
Now we need to pick a subset B of A which contains 17 members. Then the number of members of the remainder set A ~ B will be 49 - 17 .

There are many ways to choose B . One of them is this:



But when we choose B this way, the remainder set A ~ B is not easy to count. Some of the original bundles of ten have been broken up, and only pieces of them are in A ~ B .

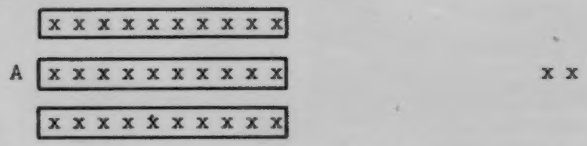
It is much better if we choose B so as to either include all of a bundle of ten or none of it. Here is one way:



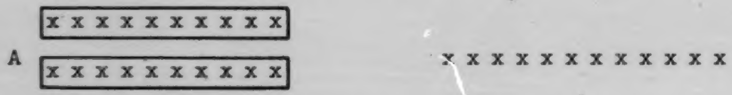
Now it is easy to count the remainder set A ~ B . It can be done in two steps. Looking at the right hand side above, we see that the number of ones in the remainder set is 9 - 7 = 2 . Looking at the left hand side above, we see that the number of bundles of ten in the remainder set is 4 - 1 = 3 . Therefore the number of members in the remainder set is 32 .

An important thing to notice is that since we dealt only with complete bundles of ten, we could count these using only "small" numbers.

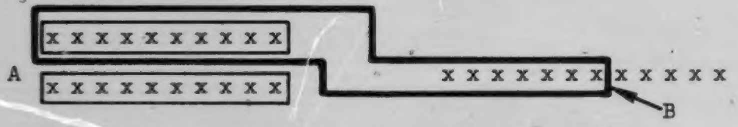
Now, let us examine in the same way another problem: 32 - 17 = n . We can pick A to be a set of 32 x's :



We need to pick a subset B with 17 members, that is, one bundle of ten and seven ones. But A has only two ones, so we will have to use some of the members of A in the bundles of ten. As we saw above, it is best if we use only whole bundles. Therefore, we will take one of the bundles of ten in A , change it to 10 ones, and put it with the 2 ones. Now A looks like this:



Now it is easy to see how we can pick a convenient subset B which has 17 members. Here is one:



It is easy to count the remainder set A ~ B . The number of ones is 12 - 7 = 5 and the number of tens is 2 - 1 = 1 . Therefore 32 - 17 = 15 , and n = 15 .

Rather than object representation we may use algorithms such as these to subtract 17 from 32 :

$$\text{(a)}\quad \begin{array}{lll} 3 \text{ tens} + 2 \text{ ones} & = & 2 \text{ tens} + 12 \text{ ones} \\ \underline{1 \text{ ten} + 7 \text{ ones}} & = & \underline{1 \text{ ten} + 7 \text{ ones}} \\ & & 1 \text{ ten} + 5 \text{ ones} = 15 \end{array}$$

or

$$\text{(b)}\quad \begin{array}{lll} 30 + 2 & = & 20 + 12 \\ \underline{10 + 7} & = & \underline{10 + 7} \\ & & 10 + 5 = 15 \end{array}$$

or we may use an equation form, as

$$\begin{aligned} 32 - 17 &= (30 + 2) - (10 + 7) \\ &= (20 + 12) - (10 + 7) \\ &= (20 - 10) + (12 - 7) \\ &= 10 + 5 \\ &= 15 \end{aligned}$$

Notice that the renaming of (30 + 2) as (20 + 12) involves an application of the associative property of addition, in that

$$(30 + 2) = ([20 + 10] + 2) = (20 + [10 + 2]) = (20 + 12) .$$

We may subtract larger numbers, of course, simply by extending the principles and procedures used with smaller numbers. Consider, for instance, subtracting 276 from 523 .

Since we cannot subtract 6 ones from 3 ones nor 7 tens from 2 tens, renaming is required. In detail, we may write:

$$\begin{aligned}
5 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} &= 5 \text{ hundreds} + (1 \text{ ten} + 1 \text{ ten}) + 3 \text{ ones}. \\
&= 5 \text{ hundreds} + (1 \text{ ten} + 10 \text{ ones}) + 3 \text{ ones}. \\
&= 5 \text{ hundreds} + 1 \text{ ten} + 13 \text{ ones}. \\
&= (4 \text{ hundreds} + 1 \text{ hundred}) + 1 \text{ ten} + 13 \text{ ones}. \\
&= (4 \text{ hundreds} + 10 \text{ tens}) + 1 \text{ ten} + 13 \text{ ones}. \\
&= 4 \text{ hundreds} + 11 \text{ tens} + 13 \text{ ones}.
\end{aligned}$$

Ordinarily this procedure is simply indicated by

$$5 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} = 4 \text{ hundreds} + 11 \text{ tens} + 13 \text{ ones}.$$

We may now complete the problem 523 - 276 by writing:

$$\begin{array}{rcl}
5 \text{ hundreds} + 2 \text{ tens} + 3 \text{ ones} & = & 4 \text{ hundreds} + 11 \text{ tens} + 13 \text{ ones} \\
\underline{2 \text{ hundreds} + 7 \text{ tens} + 6 \text{ ones}} & = & \underline{2 \text{ hundreds} + 7 \text{ tens} + 6 \text{ ones}} \\
 & & 2 \text{ hundreds} + 4 \text{ tens} + 7 \text{ ones} = 247
\end{array}$$

or we may write

$$\begin{array}{rcl}
500 + 20 + 3 & = & 400 + 110 + 13 \; . \\
\underline{200 + 70 + 6} & = & \underline{200 + 70 + 6} \\
 & & 200 + 40 + 7 = 247
\end{array}$$

or we may use an equation form, such as

$$\begin{aligned}
523 - 276 &= (500 + 20 + 3) - (200 + 70 + 6) \\
&= (400 + 110 + 13) - (200 + 70 + 6) \\
&= (400 - 200) + (110 - 70) + (13 - 6) \\
&= 200 + 40 + 7 \\
&= 247 \; .
\end{aligned}$$

We eventually may shorten such algorithms to the form

$$\begin{array}{r}
\textcircled{4}\textcircled{11}\textcircled{13} \\
5\;2\;3 \\
-\;\underline{2\;7\;6} \\
2\;4\;7
\end{array}
\qquad \text{or simply} \qquad
\begin{array}{r}
523 \\
-\;\underline{276} \\
247
\end{array}$$

Problems

3. a. In the property $(a + b) - (c + d) = (a - c) + (b - d)$, why are the conditions $a \geq c$ and $b \geq d$ needed?

   b. Give an illustration of the difficulty encountered if the conditions are not met.

4. a. Represent with an appropriate set, A, and subset, B, the subtraction of 43 and 27.

   b. Show the same subtraction in equation form.

Summary

Techniques of addition and subtraction may be explained in terms of our decimal numeration system, coupled with regrouping and applications of the commutative and associative properties of addition. Subtraction techniques utilize a special property of subtraction; namely,

If a, b, c, and d are whole numbers such that $a \geq c$ and $b \geq d$, then it is true that

$$(a + b) - (c + d) = (a - c) + (b - d) .$$

This special property may be explained in terms of the definition of subtraction in relation to addition, coupled with the commutative and associative properties of addition.

Applications to Teaching

If young children are to compute with understanding, it is essential that they have an adequate understanding of our numeration system with its base of ten and its principle of place value. They also need to have ample opportunity to manipulate sets of objects as the basis for developing appropriate algorithms.

Algorithms such as these grow readily from manipulations of sets of objects:

1. $42 + 36 = ?$

(a)	(b)	(c)
$4 \text{ tens} + 2 \text{ ones}$	$40 + 2$	42
$\underline{3 \text{ tens} + 6 \text{ ones}}$	$\underline{30 + 6}$	$+\ \underline{36}$
$7 \text{ tens} + 8 \text{ ones} = 78$	$70 + 8 = 78$	8
		$\underline{70}$
		78

2. $69 - 24 = ?$

(a)	(b)
$6 \text{ tens} + 9 \text{ ones}$	$60 + 9$
$\underline{2 \text{ tens} + 4 \text{ ones}}$	$\underline{20 + 4}$
$4 \text{ tens} + 5 \text{ ones} = 45$	$40 + 5 = 45$

These same algorithms serve young children well when regrouping and renaming are involved:

3. $58 + 17 = ?$

(a)	(b)	(c)
$5 \text{ tens} + 8 \text{ ones}$	$50 + 8$	58
$\underline{1 \text{ ten} + 7 \text{ ones}}$	$\underline{10 + 7}$	$+\ \underline{17}$
$6 \text{ tens} + 15 \text{ ones}$, or	$60 + 15$, or	15
$7 \text{ tens} + 5 \text{ ones} = 75$	$70 + 5 = 75$	$\underline{60}$
		75

4. $81 - 35 = ?$

(a)	(b)
$8 \text{ tens} + 1 \text{ one} = 7 \text{ tens} + 11 \text{ ones}$	$80 + 1 = 70 + 11$
$\underline{3 \text{ tens} + 5 \text{ ones}} = \underline{3 \text{ tens} + 5 \text{ ones}}$	$30 + 5 = \underline{30 + 5}$
$4 \text{ tens} + 6 \text{ ones} = 46$	$40 + 6 = 46$

Each child is not expected to be equally at ease with all algorithms. He should be encouraged to work with the form with which he is most comfortable. Eventually he will shorten that algorithm to a more efficient form, but he should not be hurried into doing this. Computing with understanding takes precedence over computing with a highly efficient form in the earlier stages of learning.

Exercises - Chapter 10

1. For each of these examples, compute using the three addition algorithms just illustrated in the preceding section.

   a. $246 + 139 = ?$

   b. $777 + 964 = ?$

   c. $486 + 766 = ?$

   d. $774 + 926 = ?$

2. For each of these examples, compute using the two subtraction algorithms illustrated in the preceding section.

   a. $764 - 199 = ?$

   b. $402 - 138 = ?$

   c. $710 - 287 = ?$

   d. $800 - 396 = ?$

3. Compute $774 + 926$ using an equation form.

4. Compute $800 - 396$ using an equation form.

Solutions for Problems

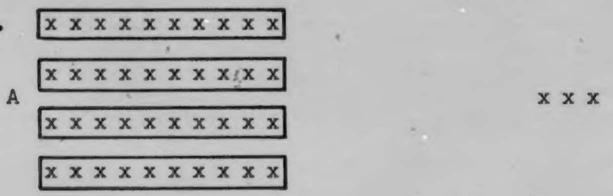
1.
$$\begin{aligned}
38 + 73 + 22 &= 38 + (73 + 22) \\
&= 38 + [(7 \text{ tens} + 3 \text{ ones}) + (2 \text{ tens} + 2 \text{ ones})] \\
&= 38 + [(7 \text{ tens} + 2 \text{ tens}) + (3 \text{ ones} + 2 \text{ ones})] \\
&= 38 + (9 \text{ tens} + 5 \text{ ones}) \\
&= 38 + 95 \\
&= (3 \text{ tens} + 8 \text{ ones}) + (9 \text{ tens} + 5 \text{ ones}) \\
&= (3 \text{ tens} + 9 \text{ tens}) + (8 \text{ ones} + 5 \text{ ones}) \\
&= 12 \text{ tens} + 13 \text{ ones} \\
&= (1 \text{ hundred} + 2 \text{ tens}) + (1 \text{ ten} + 3 \text{ ones}) \\
&= 1 \text{ hundred} + (2 \text{ tens} + 1 \text{ ten}) + 3 \text{ ones} \\
&= 1 \text{ hundred} + 3 \text{ tens} + 3 \text{ ones} \\
&= 133
\end{aligned}$$

2.
$$\begin{aligned}
(30 + 7) + (50 + 8) &= ([30 + 7] + 50) + 8 && \text{associative property} \\
&= (30 + [7 + 50]) + 8 && \text{associative property} \\
&= (30 + [50 + 7]) + 8 && \text{commutative property} \\
&= ([30 + 50] + 7) + 8 && \text{associative property} \\
&= (30 + 50) + (7 + 8) && \text{associative property}
\end{aligned}$$

3. a. In order for $a - c$ and $b - d$ to have meaning, it is necessary that $a \geq c$ and $b \geq d$. These conditions also assure that $a + b \geq c + d$ which makes $(a + b) - (c + d)$ meaningful.

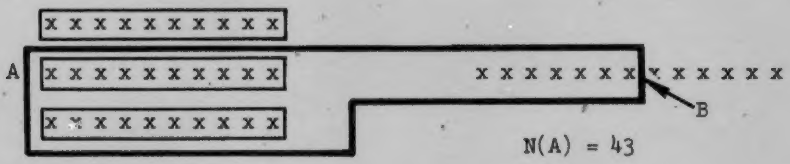
   b. For example, let $a = 7$, $b = 5$, $c = 8$, $d = 2$, so that $a \geq c$ is not true. Then $(a + b) - (c + d) = (7 + 5) - (8 + 2) = 12 - 10 = 2$, and $(a - c) + (b - d) = (7 - 8) + (5 - 2) = (7 - 8) + 3 = ?$ $7 - 8$ is not a whole number, so the property is undefined. If neither condition had been true, $(a + b) - (c + d)$ would not have been defined.

4. a.


A
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x

$N(A) = 43$

or, regrouped,


A
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x x x x
B

$N(A) = 43$

$N(B) = 27$

$N(A \sim B) = 43 - 27 = 16$

b. $43 - 27 = (40 + 3) - (20 + 7)$

$= (30 + 13) - (20 + 7)$

$= (30 - 20) + (13 - 7)$

$= 10 + 6$

$= 16$

Chapter 11

INTRODUCING RATIONAL NUMBERS

Introduction

All our work with numbers up to this point has been with the set of whole numbers; we have pretended that they are the only numbers that exist and we have seen how they and their operations behave. Our number lines have been marked only at the points which correspond to whole numbers, leaving gaps containing many points that are not named. Using only whole numbers it is clear that many division problems cannot be worked (for example $3 \div 4$); that is, the set of whole numbers is not closed under the operation of division.

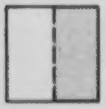
Now the problem of naming points between those named by whole numbers on the number line and the problem of (almost) getting closure under division of whole numbers (we cannot divide by zero) are two problems that persuade us of the need to extend our number system to include more than the whole numbers. In the historical development of numbers the problem of measurement (which will be considered in Chapter 12) was probably a significant motivation in forcing the extension of number systems to more sophistication than merely counting and numbering.

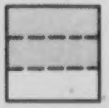
Regions as Models for Rational Numbers

In our extension of the number system to include what we will call rational numbers (but which are frequently called "fractions") we will proceed much as we did with the whole numbers. That is, we will start with physical models for such numbers and from these develop some concepts about them.

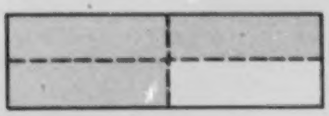
In setting up physical models for rational numbers we usually begin by designating some "basic unit", for example, a segment, a rectangular region, a circular region, or a collection of things. This unit is then partitioned into a certain number of congruent parts. These parts, compared to the unit, give us the basis for a model for rational numbers.

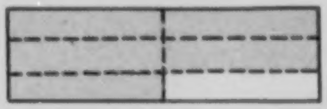
For example, let us identify as our base unit a square region and suppose this is divided into two congruent parts as shown in Figure (a). We want to associate a number with the area of the shaded part of the square. Not only do we want a number, we want a name for this number, a numeral which will remind us of the two equal parts we have, of which one is shaded. The numeral is the obvious one, $\frac{1}{2}$, read "one-half". If our unit is partitioned into three congruent parts and if two of them are shaded, as in Figure (b), the numeral $\frac{2}{3}$ reminds us that we are associating a number with two of three congruent parts of a unit. Observe that our numeral still uses notions expressible by whole numbers; that is, a basic unit is partitioned into three congruent parts with two of these considered.


(a)


(b)

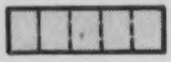
In the figures below, a rectangular region serves as the unit.


(c)

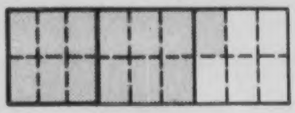

(d)

The numeral $\frac{3}{4}$ expresses the situation pictured in Figure (c), namely the unit region partitioned into four congruent regions, of which three are shaded. And, of course, the numeral $\frac{5}{6}$ expresses the situation represented by Figure (d), the base unit partitioned into six congruent regions, of which five regions are shaded.

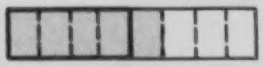
More complicated situations are represented in the next drawings. In each case the base unit is the rectangular region heavily outlined by solid lines. In some of these, the shaded region designates a region the same as or more than the base region, hence numbers equal to or than one. Thus Figure (e) shows the base unit partitioned into five parts, all of which are shaded. The numeral $\frac{5}{5}$ describes this model.



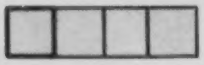
(e) Physical model for $\frac{5}{5}$



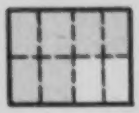
(i) $\frac{13}{6}$



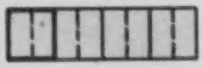
Unit
(f) Physical model for $\frac{5}{4}$



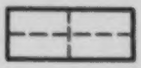
(j) $\frac{4}{1}$



(g) $\frac{6}{8}$

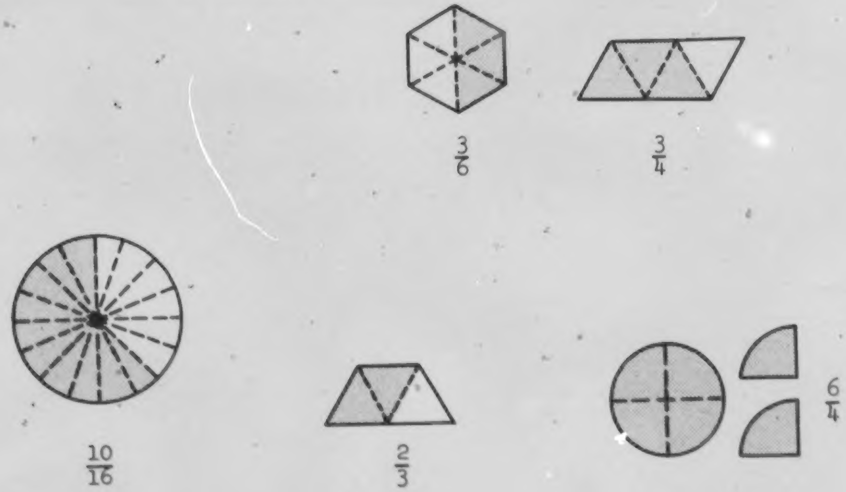


(k) $\frac{8}{2}$



(h) $\frac{0}{4}$

In Figure (f), the unit region is partitioned into four congruent regions, and five such regions are shaded; the numeral $\frac{5}{4}$ describes this model. Examine the other situations illustrated and verify that in each case the region shaded is indeed a model for the rational number named under it.



$\frac{3}{6}$ $\frac{3}{4}$

$\frac{10}{16}$ $\frac{2}{3}$ $\frac{6}{4}$

Models using regions of various shapes

Regions of other shapes can also be used as models for rational numbers. Some such regions, with associated numerals, are pictured above. In each case, you can verify that the model involves identification of a unit region, partitioning of this region into congruent regions, and consideration of a certain number of these congruent regions.

For the sake of simplicity, we have used as models only plane regions. Frequently, we use solid regions, also, as models for rational numbers. The interpretation given to such models is but an extension of that used with plane regions.

Problems*

1. Draw models for:

   a. $\frac{2}{3}$

   b. $\frac{4}{6}$

   c. $\frac{3}{2}$

   d. $\frac{12}{5}$

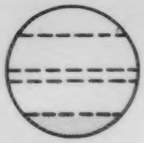
   e. $\frac{7}{7}$

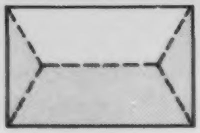
   f. $\frac{0}{6}$

---

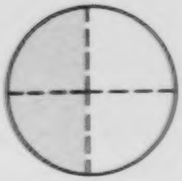
*Solutions for the problems in this chapter are on page 297 .

2. Why are the following pictures not good models for rational numbers?

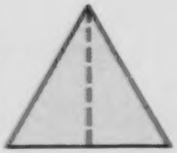




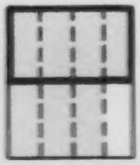
3. What numbers do the shaded portions of the following models illustrate?



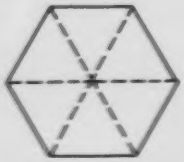
(a)



(b)



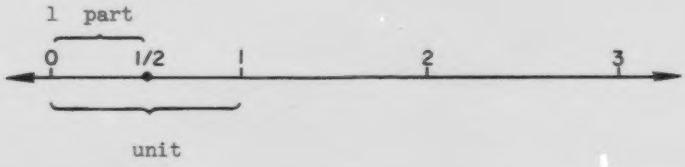
(c)



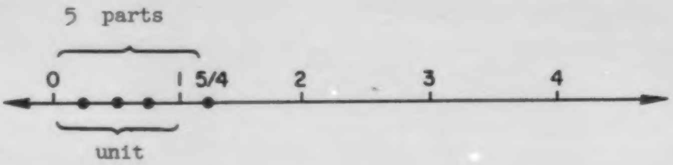
(d)

Number Line Models for Rational Numbers

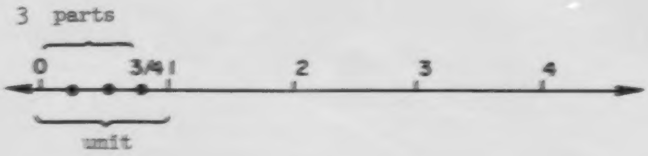
Another standard physical model for the idea of a rational number uses the number line. The way we locate new points on the number line parallels the procedure we followed with regions. After we mark off a unit segment and partition it into congruent segments, we then count these parts. Thus, in order to locate the point corresponding to $\frac{1}{2}$, we mark off the unit segment into 2 congruent parts and count off 1 of them. This point corresponds to $\frac{1}{2}$.

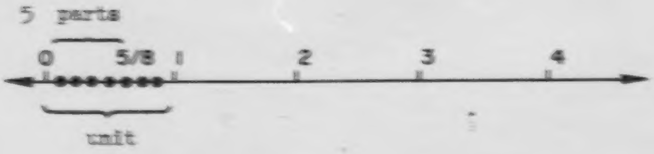


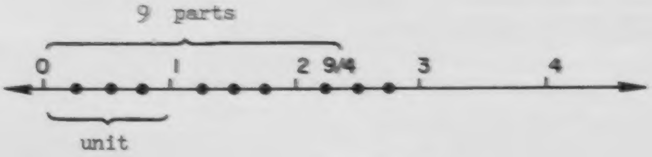
In like manner, to locate $\frac{5}{4}$, we partition a unit interval into 4 congruent parts and count off 5 of these parts. We have now located the point which we associate with $\frac{5}{4}$.


5 parts
0
1
5/4
2
3
4
unit

Once we have this method in mind, we see that we can associate a point on the number line with all symbols such as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{9}{4}$, etc., as illustrated below.


3 parts
0
3/4
1
2
3
4
unit


5 parts
0
5/8
1
2
3
4
unit


9 parts
0
1
2
9/4
3
4
unit

Problems

4. Locate the point associated with each of the following on a separate number line.

a. $\frac{0}{1}$

b. $\frac{3}{4}$

c. $\frac{3}{5}$

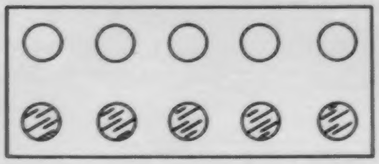
d. $\frac{5}{5}$

e. $\frac{7}{4}$

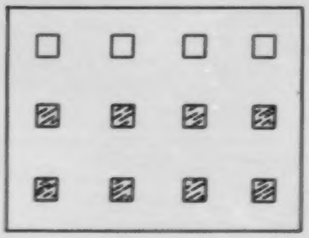
f. $\frac{8}{8}$

Array Models for Rational Numbers

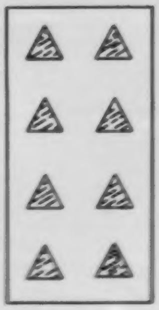
Sets of objects arranged in arrays may serve as models for rational numbers, as in the illustrations below. In each figure the unit set or array is bounded by solid lines.



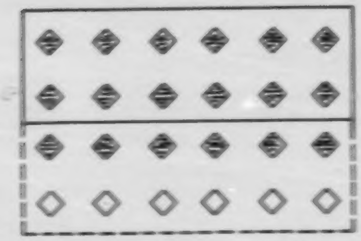
(a) A model for $\frac{1}{2}$



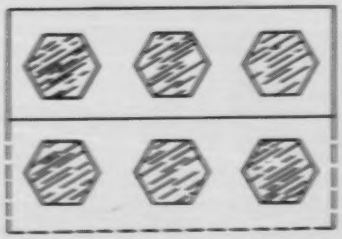
(b) A model for $\frac{2}{3}$



(c) A model for $\frac{4}{4}$

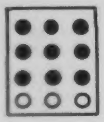


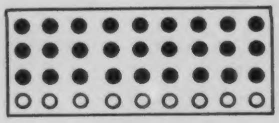
(d) A model for $\frac{3}{2}$



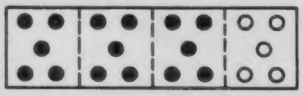
(e) A model for $\frac{2}{1}$

In Figure (a), for instance, one of the two rows of the unit array is shaded. With this model we may associate the rational number $\frac{1}{2}$. In Figure (c), four of the four rows of the unit array are shaded, and with this model we may associate the rational number $\frac{4}{4}$. There are two unit arrays in Figure (d) with two rows in each array. Three of the rows are shaded, and with this model we may associate the rational number $\frac{3}{2}$. Notice that in each instance the rational number associated with a particular model is independent of the number of elements in each row of the array. For example: we would associate the same rational number, $\frac{3}{4}$, with either of the arrays below.





Notice that we also may associate the rational number $\frac{3}{4}$ with a representation that is not an array, such as:



in which a unit set is partitioned into four equivalent subsets, three of which are to be considered.

Problems

5. Show an array as a model for each of these.

a. $\frac{5}{6}$ b. $\frac{3}{8}$ c. $\frac{7}{7}$ d. $\frac{4}{3}$ e. $\frac{7}{4}$ f. $\frac{5}{2}$

Some Vocabulary and Other Considerations

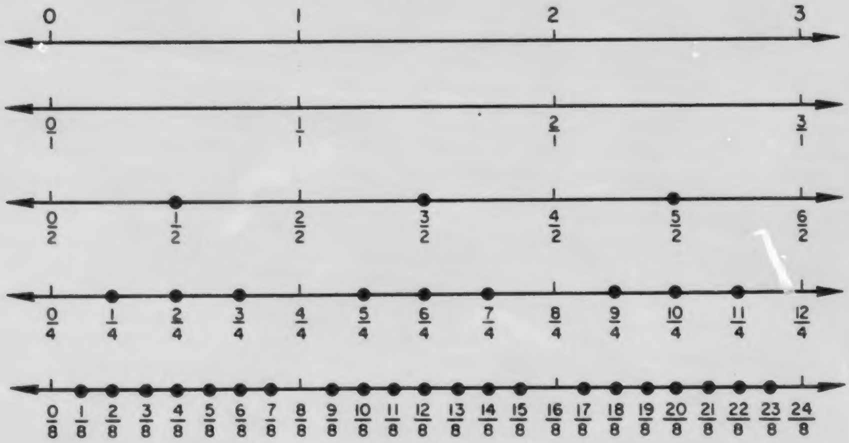
The numbers for which our regions, segments and arrays are models are called *rational numbers*. The particular numeral form in which these numbers often are expressed is called a *fraction*. Many different fractions designate the same rational number. We have here again the distinction between a number and names (numerals) for that number.

In this chapter we are concerned with those rational numbers that can be named by a fraction of the form $\frac{a}{b}$ where a represents a whole number and b represents a counting number (i.e., a whole number other than zero). In effect, this definition restricts us to a consideration of the *nonnegative* rational numbers. The complete set of rational numbers consists of those numbers of the specified form, $\frac{a}{b}$, and their opposites or negatives.

Referring to our models we see that b , the denominator, always is the number of congruent parts or equivalent subsets into which a unit has been partitioned, while a , the numerator, is the number of these congruent parts or equivalent subsets that are being used. One of several reasons why the denominator is never zero is that it would be nonsense to speak of a unit as being divided into zero parts; it surely cannot be divided into fewer than one part.

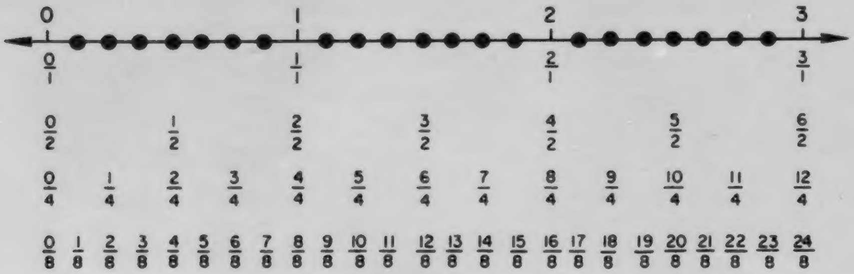
Equivalent Fractions

The following figure shows several number lines: one on which we have located points corresponding to 0 , 1 , 2 , 3 , etc.; one on which we have located points corresponding to $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.; one on which we have located points corresponding to $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc.; one on which we have located points corresponding to $\frac{0}{4}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, etc.; and one on which we have located points corresponding to $\frac{0}{8}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, etc.


0 1 2 3
0/1 1/1 2/1 3/1
0/2 1/2 2/2 3/2 4/2 5/2 6/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

As we look at these number lines, we see that it seems very natural to think of $\frac{0}{2}$, for example, as being associated with the zero point. For we are really, so to speak, counting off 0 segments. Similarly, it seems natural to locate $\frac{0}{1}$, $\frac{0}{4}$ and $\frac{0}{8}$ as indicated.

Now let us put the five number lines together, as shown in the figure below. In other words let us carry out on a single line the steps

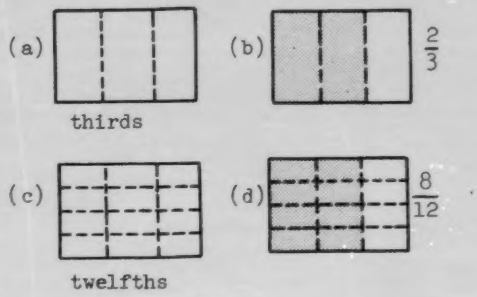


for locating in turn points corresponding to the rational numbers with denominator 1 , with denominator 2 , with denominator 4 and with denominator 8 . When we do this we see, among other things, that $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ all correspond to the same point on the number line, or, in other words, are all names (numerals) for the same rational number. We see also that $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, and so on, name the points we have formerly named with whole numbers. Furthermore we see that fractions such as $\frac{2}{2}$, $\frac{4}{4}$, $\frac{4}{2}$, $\frac{8}{4}$, and the like also name points that have formerly been named by whole numbers. Fractions which name the same point on the number line, and which therefore name the same rational number, are called equivalent fractions. Notice that corresponding to each whole number there is a set of equivalent fractions. Consequently, there is a one-to-one correspondence between the set of whole numbers and a particular subset of the set of rational numbers. Furthermore, it can be shown that a one-to-one correspondence may be established between the set of whole numbers and the entire set of rationals.

Equivalent Fractions in "Higher Terms"

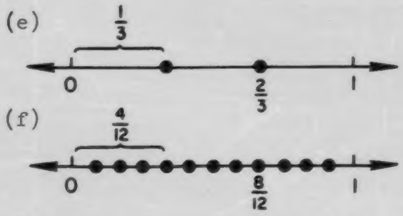
Recognizing the same rational number under a variety of disguises (names) and being able to change the names of numbers without changing the numbers are great conveniences in operating efficiently with rational numbers. Such an addition problem as $\frac{1}{4} + \frac{2}{3}$ is certainly worked out most efficiently by considering the equivalent problem $\frac{3}{12} + \frac{8}{12}$, equivalent because $\frac{1}{4}$ names the same number as $\frac{3}{12}$ and $\frac{2}{3}$ names the same number as $\frac{8}{12}$.

The figures illustrate a way of using our unit region model to show that $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions, that is, that $\frac{2}{3}$ and $\frac{8}{12}$ name the same number. First we select a unit region and partition it into three congruent regions by vertical lines as shown in Figure (a). Figure (b) shows the shading of two of these regions to represent $\frac{2}{3}$. If we return now to our unit region and partition each of the former three congruent parts by horizontal lines into four congruent parts, we have the unit partitioned into $3 \times 4 = 12$ congruent parts, as shown in Figure (c) If the unit partitioned in this way is now superimposed on the model for $\frac{2}{3}$, we get the model shown in Figure (d), which shows that each of the two shaded regions in the model for $\frac{2}{3}$ is partitioned into four regions, giving $2 \times 4 = 8$ smaller congruent regions shaded. Hence the model showing 8 of 12 congruent parts represents the same number as the model showing 2 of 3 congruent parts.



Model showing $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$.

The number lines in Figures (e) and (f) demonstrate this same equivalence. In Figure (e), $\frac{2}{3}$ is shown by partitioning the unit segment into 3 congruent parts and using two of these to mark a point. If each of the 3 congruent parts of the unit is now partitioned into 4 congruent parts,



Number line model showing that $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$.

the unit segment then contains $3 \times 4 = 12$ parts while the 2 original parts used to mark $\frac{2}{3}$ now contain $2 \times 4 = 8$ congruent parts, as shown in Figure (f). Hence, the same point is named by $\frac{8}{12}$ as was formerly named by $\frac{2}{3}$.

To put this in more general terms, consider the fraction $\frac{a}{b}$ where b represents the number of parts a unit has been partitioned into and a the number of these parts marked in the model. If each of the b parts is further partitioned into k congruent parts, the unit then contains $b \times k$ congruent parts. At the same time, each of the a parts is further partitioned into k parts so that there will be $a \times k$ smaller congruent parts marked in the model. Hence, $\frac{a \times k}{b \times k}$ represents the same number as $\frac{a}{b}$ formerly did. Symbolically:

$$\frac{a}{b} = \frac{a \times k}{b \times k}$$

where k represents any counting number. Hence, for instance,

$$\frac{3}{4} = \frac{3 \times 2}{4 \times 2} = \frac{6}{8}\text{, or } \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}\text{, or } \frac{3}{4} = \frac{3 \times 4}{4 \times 4} = \frac{12}{16}\text{, etc.}$$

Our knowledge of multiples of numbers can be used to good advantage when each of two fractions such as $\frac{5}{6}$ and $\frac{3}{4}$ is to be changed to "higher terms" so that each fraction has the same denominator.

The set of multiples of 6 is {6 , 12 , 18 , 24 , 30 , 36 , . . .} .

The set of multiples of 4 is {4 , 8 , 12 , 16 , 20 , 24 , . . . } .

The intersection of these two sets is {12 , 24 , 36 , 48 , . . .} and any member of this intersection can serve as the "common denominator" for the new fractions. The least common denominator would be 12 , of course, so that

$$\frac{5}{6} = \frac{5 \times 2}{6 \times 2} = \frac{10}{12} \quad\text{and}\quad \frac{3}{4} = \frac{3 \times 3}{4 \times 3} = \frac{9}{12}\,.$$

Problems

6. Draw both a unit region model and a number line model to illustrate that $\frac{2}{3} = \frac{4}{6}$.

7. Supply the missing numbers in each of the following.

   a. $\frac{3}{5} = \frac{3 \times \_\_}{5 \times \_\_} = \frac{24}{40}$ b. $\frac{7}{8} = \frac{\_\_}{32}$ c. $\frac{\_\_}{12} = \frac{14}{24}$

8. Specify the "k" used in each case to change the first fraction to the second.

   a. $\frac{7}{13} = \frac{7 \times k}{13 \times k} = \frac{28}{42}$; k = __

   b. $\frac{14}{16} = \frac{42}{48}$; k = __

   c. $\frac{3}{7} = \frac{63}{147}$; k = __

Equivalent Fractions in "Lower Terms"

Expressing a fraction in "lower terms" (often called "reducing" fractions) is simply reversing, or undoing, the process used to express fractions in "higher terms". For example, $\frac{2}{3} = \frac{2 \times 10}{3 \times 10} = \frac{20}{30}$ and, undoing this process, $\frac{20}{30} = \frac{20 \div 10}{30 \div 10} = \frac{2}{3}$. Similarly, $\frac{10}{4} = \frac{10 \div 2}{4 \div 2} = \frac{5}{2}$, $\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$; $\frac{147}{3} = \frac{147 \div 3}{3 \div 3} = \frac{49}{1}$ and so on. In general:

> If a counting number, k, is a factor of both a and b, then $\frac{a}{b} = \frac{a \div k}{b \div k}$.

In this case we say that the fraction $\frac{a}{b}$ has been changed to "lower terms". It should be noted that while it is always possible to change a fraction to an equivalent one in "higher terms" with denominator any desired multiple of the original denominator, it is not always possible to rename ("reduce") a fraction using a specified divisor (factor), since one cannot always divide a counting number by a counting number. For example, $\frac{4}{6}$ can be renamed using 2 as a divisor, but not by using 3, while $\frac{3}{5}$ cannot be changed to any "lower terms". We sometimes say that a fraction which cannot be changed to any "lower terms", such as $\frac{1}{3}$, $\frac{4}{7}$, etc., is in simplest form or lowest terms.

Putting fractions in lowest terms or simplest form is a convenient skill, but its importance has been overrated. The superstition that fractions must always, ultimately, be written in this form has no mathematical basis, for only different names for the same number are at issue. It is often convenient for purposes of further computation or to make explicit a particular interpretation to leave results in other than simplest form. However, where simplest form is desired we can proceed by repeated division in both numerator and denominator, or we can use the greatest common factor of both numerator and denominator as the k by which both should be divided. The greatest common factor of two numbers is the greatest whole number which is a factor of both numbers and this is precisely what is required. The examples displayed below should be sufficient to illustrate both procedures for writing a fraction in simplest form.

(a) $\frac{12}{20} = \frac{12 \div 2}{20 \div 2} = \frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$

$12 = (2 \times 2) \times 3$

$20 = (2 \times 2) \times 5$

So the greatest common factor of 12 and 20 is the "common block" of factors $2 \times 2 = 4$, and

$\frac{12}{20} = \frac{12 \div 4}{20 \div 4} = \frac{3}{5}$.

(b) $\frac{104}{260} = \frac{104 \div 2}{260 \div 2} = \frac{52}{130} = \frac{52 \div 2}{130 \div 2} = \frac{26}{65} = \frac{26 \div 13}{65 \div 13} = \frac{2}{5}$

$$\begin{array}{r|l} 2 & 104 \\ \hline 2 & 52 \\ \hline 2 & 26 \\ \hline & 13 \end{array} \qquad\qquad \begin{array}{r|l} 2 & 260 \\ \hline 2 & 130 \\ \hline 5 & 65 \\ \hline & 13 \end{array}$$

So the greatest common factor is the "common block" $2 \times 2 \times 13 = 52$, and

$\frac{104}{260} = \frac{104 \div 52}{260 \div 52} = \frac{2}{5}$.

Observe that for a fraction such as $\frac{2}{9}$ the greatest common factor of 5 and 9 is one, and consequently the fraction already is in its lowest terms. It is true that $\frac{5}{9} = \frac{5 \div 1}{9 \div 1} = \frac{5}{9}$, but there is no need to perform such a division.

Problems

9. For each of the following, give one equivalent fraction in "higher terms" and give three equivalent fractions in "lower terms", including one in lowest terms.

   a. $\frac{24}{36}$ b. $\frac{30}{60}$

10. Why would it not make sense to speak of a fraction raised to "highest terms"?

11. For each of the following, specify the greatest common factor, say f, of the numerator and denominator and use f to write the fraction in simplest form.

    a. $\frac{30}{45}$ f = _____ $\frac{30}{45} =$

    b. $\frac{24}{36}$ f = _____ $\frac{24}{36} =$

    c. $\frac{39}{52}$ f = _____ $\frac{39}{52} =$

Equality and Order Among Rational Numbers

First let us recall the three possible relations that may exist between two whole numbers, m and n. One and only one of these three things is true:

$m = n$ (m is equal to n)
$m > n$ (m is greater than n)
$m < n$ (m is less than n)

A similar statement can be made about two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$:

$\frac{a}{b} = \frac{c}{d}$ ($\frac{a}{b}$ is equal to $\frac{c}{d}$)
$\frac{a}{b} > \frac{c}{d}$ ($\frac{a}{b}$ is greater than $\frac{c}{d}$)
$\frac{a}{b} < \frac{c}{d}$ ($\frac{a}{b}$ is less than $\frac{c}{d}$)

Let us consider these three specific examples:

1. $\frac{6}{8}, \frac{9}{12}$ 2. $\frac{7}{8}, \frac{5}{6}$ 3. $\frac{5}{8}, \frac{4}{6}$

How may we compare the rational numbers in each example to determine whether the first number of each pair is equal to, or greater than, or less than the second number of each pair? Of the several approaches that might be taken, we shall illustrate the one in which each pair of fractions is expressed in terms of equivalent fractions whose denominators are the same. In particular, the common denominator will be the least common denominator. Thus:

1. To compare $\frac{6}{8}$ and $\frac{9}{12}$: since $\frac{6}{8} = \frac{18}{24}$, $\frac{9}{12} = \frac{18}{24}$, and $\frac{18}{24} = \frac{18}{24}$ it must be true that $\frac{6}{8} = \frac{9}{12}$
2. To compare $\frac{7}{8}$ and $\frac{5}{6}$: since $\frac{7}{8} = \frac{21}{24}$, $\frac{5}{6} = \frac{20}{24}$, and $\frac{21}{24} > \frac{20}{24}$, it must be true that $\frac{7}{8} > \frac{5}{6}$.
3. To compare $\frac{5}{8}$ and $\frac{4}{6}$: since $\frac{5}{8} = \frac{15}{24}$, $\frac{4}{6} = \frac{16}{24}$, and $\frac{15}{24} < \frac{16}{24}$, it must be true that $\frac{5}{8} < \frac{2}{3}$.

Now let us summarize each of these three comparisons and also make a significant observation in each instance:

1. $\frac{6}{8} = \frac{9}{12}$. It also is true that $6 \times 12 = 8 \times 9$.
2. $\frac{7}{8} > \frac{5}{6}$. It also is true that $7 \times 6 > 8 \times 5$.
3. $\frac{5}{8} < \frac{2}{3}$. It also is true that $5 \times 3 < 8 \times 2$.

It is extremely dangerous to generalize on the basis of isolated examples! However, the preceding examples do illustrate an important set of relations that can be demonstrated to be true for all nonnegative rational numbers $\frac{a}{b}$ and $\frac{c}{d}$:

$\frac{a}{b} = \frac{c}{d}$ if and only if $a \times d = b \times c$.

$\frac{a}{b} > \frac{c}{d}$ if and only if $a \times d > b \times c$.

$\frac{a}{b} < \frac{c}{d}$ if and only if $a \times d < b \times c$.

Thus, we have a very simple and convenient way for determining whether or not two rational numbers are equal and, if not equal, a very simple and convenient way for ordering them.

Problem

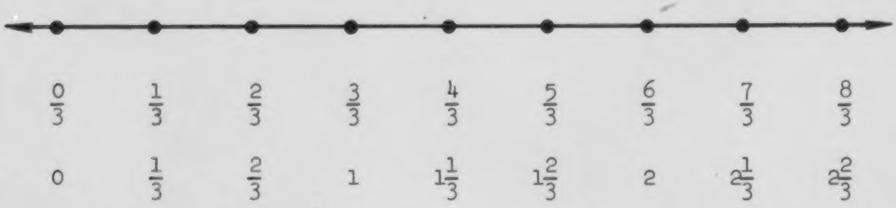
12. Make each of the following statements true by writing $=$ or $>$ or $<$ in the ring.

a. $\frac{6}{14} \bigcirc \frac{7}{16}$ b. $\frac{6}{8} \bigcirc \frac{9}{12}$ c. $\frac{30}{63} \bigcirc \frac{15}{28}$

d. $\frac{3}{4} \bigcirc \frac{36}{52}$ e. $\frac{9}{20} \bigcirc \frac{45}{100}$ f. $\frac{143}{13} \bigcirc \frac{1043}{103}$

Rational Numbers in Mixed Form

Each of us is familiar with the fact that a rational number whose name is $\frac{3}{2}$, for example, also may be named in the mixed form, $1\frac{1}{2}$. (We prefer to speak of the mixed form for a rational number rather than to speak of a "mixed number".) Let us use the number line to examine briefly some of the assumptions underlying our use of the familiar mixed form for naming certain rational numbers.



Consider, for instance, the use of $\frac{5}{3}$ and $1\frac{2}{3}$ to name the same rational number. We often state that $\frac{5}{3} = 1\frac{2}{3}$. Behind this statement there is the assumption, among others, that rational numbers can be added: $\frac{5}{3} = \frac{3}{3} + \frac{2}{3} = 1 + \frac{2}{3} = 1\frac{2}{3}$.

Or, consider the statement that $\frac{7}{3} = 2\frac{1}{3}$. Here again we see that the ability to add rational numbers is one of the things underlying our interpretation of $2\frac{1}{3}$, since: $\frac{7}{3} = \frac{6}{3} + \frac{1}{3} = 2 + \frac{1}{3} = 2\frac{1}{3}$.

It is beyond the scope of this chapter to give any systematic consideration to the addition of rational numbers. However, we did wish to point out that this operation is implicit in an interpretation of the mixed form for a rational number.

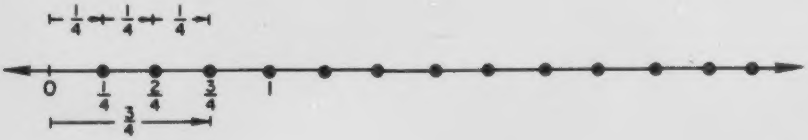
Another important implicit assumption is considered in the following section.

Rational Numbers and Division

Thus far rational numbers have been interpreted in terms of several models: unit regions partitioned into congruent regions, unit sets or arrays partitioned into equivalent subsets, and unit segments partitioned into congruent segments. We shall now look more closely at the interpretation of rational numbers on the number line.

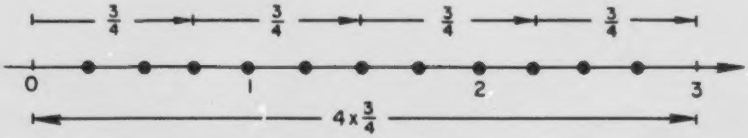
For an example we shall consider $\frac{3}{4}$. We partition the unit segment into four congruent subsegments and count three of them. Each interval in the partition represents $\frac{1}{4}$, therefore three-fourths is the union of three of these subsegments. Numerically this implies that $\frac{3}{4}$ is defined as $3 \times \frac{1}{4}$.

(a)



Similarly, the union of four of these segments abutted end-to-end represents $4 \times \frac{3}{4}$ or 3, as shown in (b).

(b)



This is consistent with the above definition and the associative property of multiplication for the product:

$$4 \times \frac{3}{4} = 4 \times \left(3 \times \frac{1}{4}\right) = (4 \times 3) \times \frac{1}{4} = 12 \times \frac{1}{4} = \frac{12}{4} = 3 .$$

The equality of the first and last numerals are of particular interest:

$$4 \times \frac{3}{4} = 3$$

It demonstrates that there is a number, n, that satisfies the equation

$$4 \times n = 3 ,$$

namely, $n = \frac{3}{4}$. Associated with this equation is the quotient $n = 3 \div 4$. This had no meaning in the set of whole numbers, but we see now that the set of rational numbers provides the number $\frac{3}{4}$ as equal to $3 \div 4$.

Recall the use of the number line in illustrating division, say of $6 \div 3$. A 6 unit segment is partitioned into 3 congruent subsegments. Each subsegment is congruent to the segment from 0 to 2, and thus, $6 \div 3 = 2$. A similar partitioning of a 3 unit segment into 4 congruent subsegments can be associated with $3 \div 4$. As Figure (b) above shows, each subsegment is congruent to the segment from 0 to $\frac{3}{4}$, thus justifying further that $3 \div 4 = \frac{3}{4}$.

This is but one illustration of an important relation between rational numbers and division. In general, it is true that

$$a \div b = \frac{a}{b}$$

where a is any whole number, b is any counting number, and their quotient is the rational number $\frac{a}{b}$. Thus, for every whole number a and for every counting number b there is a rational number $\frac{a}{b}$ such that

$$b \times \frac{a}{b} = a .$$

Problem

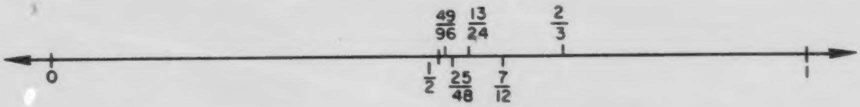
13. a. Find n if $3 \times n = 5$.

    b. Show the division on the number line.

A New Property of Numbers

Rational numbers are different in many ways from whole numbers. One such difference is apparent if we recall that for any whole number one can always say what the "next" whole number is and then ask, in a similar vein, what the "next" rational number is after any given rational number. For example, 4 is the next whole number after 3 , 1069 is the next whole number after 1068 , and so on. What is the next rational number after $\frac{1}{2}$? If $\frac{2}{3}$ is suggested as the next one, we can observe that $\frac{1}{2} = \frac{6}{12}$ and $\frac{2}{3} = \frac{8}{12}$, so $\frac{7}{12}$ is surely between $\frac{1}{2}$ and $\frac{2}{3}$. Hence, $\frac{7}{12}$ has a better claim to being next to $\frac{1}{2}$ than does $\frac{2}{3}$. If it is then suggested that $\frac{7}{12}$ be regarded as the next number after $\frac{1}{2}$, we can observe that $\frac{1}{2} = \frac{12}{24}$ and $\frac{7}{12} = \frac{14}{24}$ so $\frac{13}{24}$ is closer to $\frac{1}{2}$ than is $\frac{7}{12}$. To carry this one step further, we can squelch anyone who suggests $\frac{13}{24}$ as being the next number after $\frac{1}{2}$ by pointing out that $\frac{1}{2} = \frac{24}{48}$ and $\frac{13}{24} = \frac{26}{48}$ so that $\frac{25}{48}$ is more nearly "next to" $\frac{1}{2}$ than is $\frac{13}{24}$. It is clear that this process could be carried on indefinitely and, furthermore, would apply no matter what rational number was involved. That is, we can never identify a "next" rational number after any given rational number. A similar argument would show that we cannot identify a number "just before" a given rational number.

A number line with a very large unit is shown to illustrate the process we went through in searching for the number "next to" $\frac{1}{2}$.

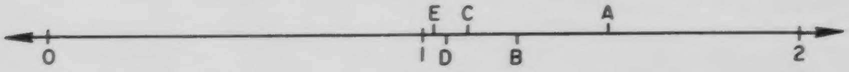


Another way of expressing what we have been talking about is to say that between any two rational numbers, there is always a third rational number; in fact, there are more rational numbers than we could count. Mathematicians sometimes describe this by saying that the set of rational numbers is dense. The word is not important to us, but is descriptive of the packing of points representing rational numbers closer and closer together on the number line. Although we can visualize that the points representing the rational numbers are densely packed, there are many points on the number line whose coordinates are not rational numbers.

Many points are associated with numbers such as π , $\sqrt{2}$, $\sqrt[3]{7}$, and so on. We are not going to consider such numbers in this text, but we mention them to indicate that the number line is not yet complete. There is a point associated with every rational number but there is not a rational number for every point.

Problems

14. Name the rational numbers associated with the points A , B , C , D and E below, where A is halfway between 1 and 2 , B halfway between 1 and A , etc.



15. How many numbers are there between 1 and the number associated with point E ?

Summary

Every nonnegative rational number can be represented by many different fractions of the form $\frac{a}{b}$, where a designates a whole number and b designates a counting number. All fractions for the same rational number are said to be equivalent. The problems of changing a fraction to "higher terms" or to "lower terms" or to lowest terms are essentially problems of renaming. In this connection we use to advantage the fact that

$$\frac{a}{b} = \frac{a \times k}{b \times k} \quad \text{(where } k \text{ designates a counting number)}$$

and also the fact that

$$\frac{a}{b} = \frac{a \div k}{a \div k} \quad \text{(where } k \text{ designates a factor of } a \text{ and } b \text{).}$$

Equality and order among the nonnegative rational numbers can be established on the basis of these conditions:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad a \times d = b \times c \ .$$

$$\frac{a}{b} > \frac{c}{d} \quad \text{if and only if} \quad a \times d > b \times c \ .$$

$$\frac{a}{b} < \frac{c}{d} \quad \text{if and only if} \quad a \times d < b \times c \ .$$

We have seen that a rational number may be used to designate the quotient of any whole number, a , and any counting number, b :

$$a \div b = \frac{a}{b} \ .$$

Finally, we have pointed to the fact that between any two rational numbers, no matter how close they are to each other, there are many other rational numbers. Among other things this means that, unlike the whole numbers, one cannot identify the number that comes "just before" or "just after" a given rational number.

Applications to Teaching

We have emphasized the use of several different models in developing ideas about rational numbers:

a. unit regions (plane and solid), partitioned into congruent regions;

b. unit segments, partitioned into congruent segments; and

c. unit arrays (or sets), partitioned into equivalent subsets.

Children encounter each of these models in connection with their everyday experiences, such as:

a. displaying a fractional part of a candy bar,
b. displaying a fractional part of a piece of string,
c. displaying a fractional part of a bag of marbles.

It is important that children have ample experience with each of the models identified if children are to be able to apply rational numbers correctly and effectively. Variety of representation is imperative in this connection.

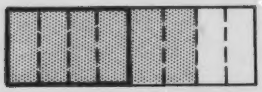
1. Using rectangular regions as your unit regions, represent each of the following by partitioning the units and shading in parts.

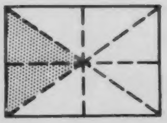
a.	$\frac{3}{4}$	e.	$\frac{7}{5}$
b.	$\frac{2}{4}$	f.	$\frac{0}{3}$
c.	$\frac{4}{4}$	g.	$\frac{9}{4}$
d.	$\frac{5}{4}$	h.	$\frac{1}{7}$

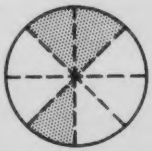
2. Using unit segments on number lines, represent each of the fractions a - h of Exercise 1.

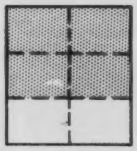
3. Using arrays or equivalent sets, represent each of the fractions a - h of Exercise 1.

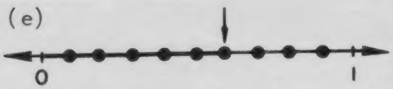
4. Most of the following figures are models for rational numbers. Some of them are not models because the unit has not been partitioned into congruent parts. For each one that is a proper model, give the rational number which is pictured.

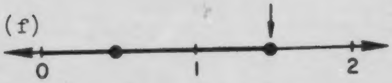

(a)

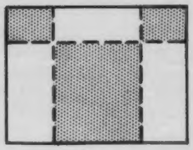

(b)

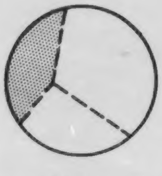

(c)

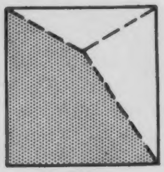

(d)

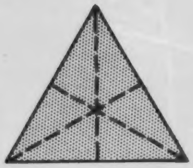
(e)


(f)


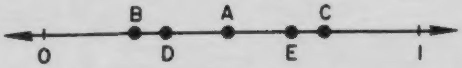

(g)


(h)


(i)


(j)

5. Consider the points labeled A , B , C , D and E on the number line:



   a. Give a fraction name to each of the points.
   b. Is the rational number located at point B less than or greater than the one located at D ? Explain your answer.
   c. In terms of the marks on this number line, what two fraction names could be assigned to the point A ?

6. Interpret on the number line the following:

   a. $\frac{20}{5} = 4$ b. $\frac{20}{4} = 5$ c. $\frac{23}{5} = 4\frac{3}{5}$

7. Show on the number line the equality:

$$\frac{2}{8} = \frac{3}{12}$$

8. Tell which of the following fractions are in "simplest form".

$$\frac{6}{12}, \frac{11}{4}, \frac{7}{12}, \frac{1a}{13}, \frac{510}{513}, \frac{7}{412}, \frac{412}{7}, \frac{10}{12}, \frac{13}{26}, \frac{2}{3}$$

9. For each pair of rational numbers named below, indicate whether the first is equal to the second, greater than the second, or less than the second.

   a. $\frac{1}{25}, \frac{1}{24}$ c. $\frac{7}{8}, \frac{5}{6}$ e. $\frac{13}{26}, \frac{9}{18}$

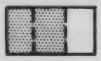
   b. $\frac{11}{24}, \frac{12}{26}$ d. $\frac{17}{32}, \frac{1}{2}$

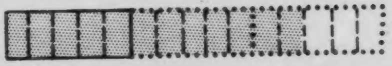
10. Express each of these in mixed form.

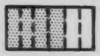
   a. $\frac{7}{4}$ b. $\frac{15}{8}$ c. $\frac{21}{9}$ d. $\frac{34}{15}$ e. $\frac{56}{12}$

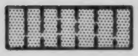
Solutions for Problems

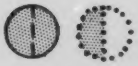
1. Many models may be used. These are illustrative only.

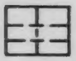
a.  $\frac{2}{3}$

d.  $\frac{12}{5}$

b.  $\frac{4}{6}$

e.  $\frac{7}{7}$

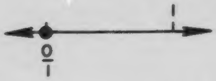
c.  $\frac{3}{2}$

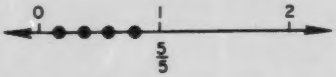
f.  $\frac{0}{6}$

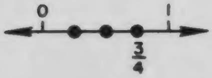
2. The figures are not good models because they are not partitioned into congruent regions.

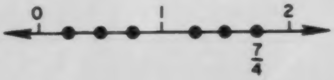
3. a. $\frac{2}{4}$ b. $\frac{2}{2}$ c. $\frac{7}{4}$ d. $\frac{0}{6}$

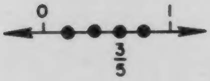
4.

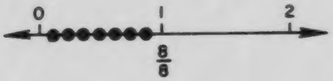
a. 

d. 

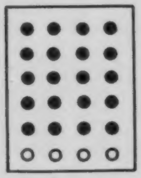
b. 

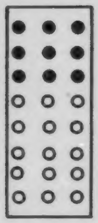
e. 

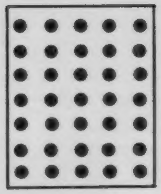
c. 

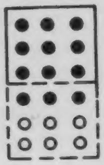
f. 

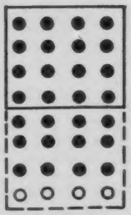
5. These models are illustrative only.

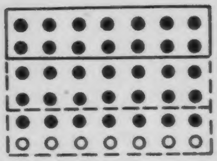
a. 

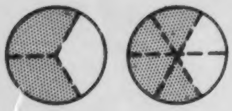
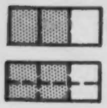
b. 

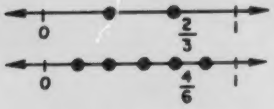
c. 

d. 

e. 

f. 

6. For example:  or 

or 
0
$\frac{2}{3}$
1
0
$\frac{4}{6}$
1

7. a. 8 b. 28 c. 7

8. a. $k = 4$ b. $k = 3$ c. $k = 21$

9.

	Higher terms; many answers, e.g.:	Lower terms; any of these:	Lowest terms:
a.	$\frac{48}{72}$, $\frac{72}{108}$, $\frac{240}{360}$, etc.	$\frac{12}{18}$, $\frac{8}{12}$, $\frac{6}{9}$, $\frac{4}{6}$, $\frac{2}{3}$	$\frac{2}{3}$
b.	$\frac{60}{120}$, $\frac{180}{240}$, $\frac{240}{480}$, etc.	$\frac{15}{30}$, $\frac{10}{20}$, $\frac{6}{12}$, $\frac{5}{10}$, $\frac{3}{6}$, $\frac{1}{2}$	$\frac{1}{2}$

10. Since in $\frac{a}{b} = \frac{a \times k}{b \times k}$ $\underline{k}$ can be any counting number, there is no limit to how large the numerator and denominator can become.

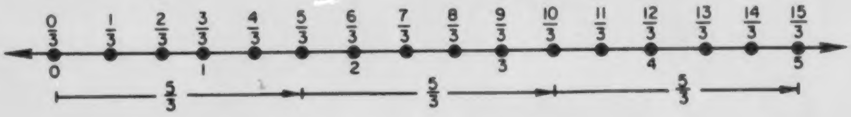
11. a. $f = 15$, $\frac{30 \div 15}{45 \div 15} = \frac{2}{3}$ c. $f = 13$, $\frac{39 \div 13}{52 \div 13} = \frac{3}{4}$

b. $f = 12$, $\frac{24 \div 12}{36 \div 12} = \frac{2}{3}$

12. a. $\frac{6}{14} \textcircled{<} \frac{7}{16}$ b. $\frac{6}{8} \textcircled{=} \frac{9}{12}$ c. $\frac{30}{63} \textcircled{<} \frac{15}{28}$

d. $\frac{3}{4} \textcircled{>} \frac{36}{52}$ e. $\frac{9}{20} \textcircled{=} \frac{45}{100}$ f. $\frac{143}{13} \textcircled{>} \frac{1043}{103}$

13. a. $n = 5 \div 3 = \frac{5}{3}$

b. 
$\frac{0}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
$\frac{7}{3}$
$\frac{8}{3}$
$\frac{9}{3}$
$\frac{10}{3}$
$\frac{11}{3}$
$\frac{12}{3}$
$\frac{13}{3}$
$\frac{14}{3}$
$\frac{15}{3}$
0
1
2
3
4
5
$\frac{5}{3}$
$\frac{5}{3}$
$\frac{5}{3}$

14.

A	B	C	D	E
$\frac{3}{2}$ (or $1\frac{1}{2}$)	$\frac{5}{4}$ (or $1\frac{1}{4}$)	$\frac{9}{8}$ (or $1\frac{1}{8}$)	$\frac{17}{16}$ (or $1\frac{1}{16}$)	$\frac{33}{32}$ (or $1\frac{1}{32}$)

15. More than can be counted (actually "infinitely many").

Chapter 12

MEASUREMENT

Introduction

Measurement is one of the connecting links between the physical world around us and mathematics. So is counting, but in a different way. We count the number of books on the desk, but measure the length of the desk. Measurement is also a connecting link between numbers and geometric figures. To measure a line segment is to assign a number to it. This cannot be done by counting the points of the segment since there are infinitely many points in any segment. To take the place of counting the points, some new concept must be developed. The concept of "measurement" that will be developed is applicable not only to line segments but in a closely related fashion to angles, areas of regions, volumes of solids, weight, time, work, energy, and many other concepts or physical entities.

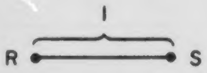
The Measure of a Segment

In mathematics we think of the endpoints of a line segment as being exact locations in space. The line segment determined by these endpoints is considered to have a certain exact length. For instance, the endpoints A and B of $\overline{AB}$ are exact locations in space, and $\overline{AB}$ itself has an exact length as one of its properties. Exact length, then, is a property of all segments. In our intuitive concept of congruence, we have said that two segments are congruent if a movable copy of one can be "matched and fitted exactly" on the other. This may be interpreted as meaning that the two segments have the same length. Thus, the common property of congruent segments is the same length. Non-congruent segments have different lengths which enable them to be ordered. When we compare $\overline{AB}$ with any other segment such as $\overline{CD}$, one and only one of these three things is true:

$\overline{AB}$ is longer than $\overline{CD}$, or
$\overline{AB}$ is exactly as long as $\overline{CD}$, or
$\overline{AB}$ is shorter than $\overline{CD}$.

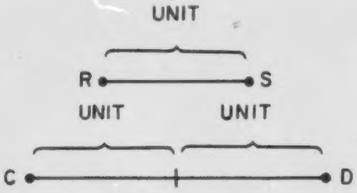
In the case of finite sets, examination revealed a property on the basis of which the sets could be compared. That is, one set could match a second set or it could have more or fewer members than the second set. At that point, numbers were associated with the property. In the same way, we wish to associate numbers with the property of length of segments. This is the objective of measurement, or finding the length of a segment.

Let us describe the process of measurement as it applies to line segments. The first step is to choose a line segment, say $\overline{RS}$, to serve as one unit. This means to select $\overline{RS}$ and agree to consider its measure to be exactly the number 1 .



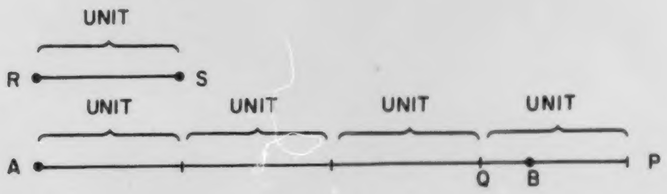
(We should recognize that this selection of a unit is an arbitrary choice we make. Different people might well choose different units and historically they have, giving rise to much confusion. For example, at one time the English "foot" was actually the length of the foot of the reigning king and the "yard" the distance from his nose to the end of his outstretched arm. Imagine the confusion when the king died if the next one was of much different stature. Various standard units will be discussed a little later but meanwhile we return to the choice of $\overline{RS}$ as our unit, recognizing the arbitrariness of this choice.)

Now it is possible to conceive of a line segment, $\overline{CD}$, such that the unit $\overline{RS}$ can be laid off exactly twice along $\overline{CD}$, as suggested in the next drawing.



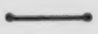
Then by agreement the measure of $\overline{CD}$ is the number 2 and the length of $\overline{CD}$ is exactly 2 units, although $\overline{CD}$ can be represented only approximately by a drawing. In the same way, line segments of length exactly 3 units, or exactly 4 units, or exactly any larger number of units are conceptually possible, although such line segments can be drawn only approximately. In fact, if a line segment is very long -- say a million inches long -- no one would want to try to draw it even approximately; but such a segment can still be thought of.

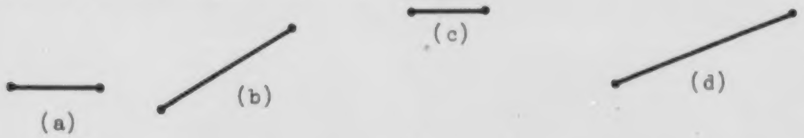
We can also conceive of a line segment, $\overline{AB}$, such that the unit $\overline{RS}$ will not "fit into" $\overline{AB}$ a whole number of times at all. $\overline{AB}$ is a line segment such that starting at A the unit $\overline{RS}$ can be laid off 3 times along $\overline{AB}$ reaching Q which is between A and B , although if it were laid off 4 times we would arrive at a point P which is well beyond B . What can be said about the length of $\overline{AB}$? Well, surely $\overline{AB}$ has length greater than 3 units and less than 4 units. In this particular case, we can also estimate visually that the length of $\overline{AB}$ is nearer to 3 units than to 4 units, so that to the nearest unit the length of $\overline{AB}$ is 3 units. This is the best we can do without considering fractional parts of units, or else shifting to a smaller unit.

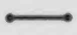


Another way of describing length to the nearest unit is by using the word "measure". Thus the measure of $\overline{AB}$, denoted $m(\overline{AB})$, is the number 3 . It is understood in the use of measure that it does not necessarily describe exact length. If two segments have the same length, we know they are congruent and they have the same measure. Two segments with the same measure in terms of a specified unit are not necessarily congruent. However, if two segments have the same measure for every specified unit, no matter how small, they must be congruent.

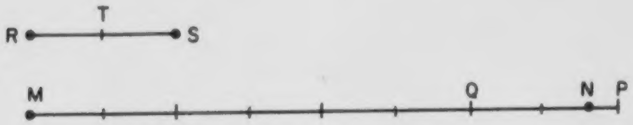
Problems*

1. Using the unit  find the measure of each of the following segments to the nearest unit.



2. Using the unit  find the measure of each of the segments in Problem 1 to the nearest unit.

To help us in estimating whether the measure of a segment is say, 3 or 4 , we need to bisect our unit. $\overline{RS}$ is again shown as our unit with T bisecting $\overline{RS}$ so that $\overline{RT}$ is congruent to $\overline{TS}$ and $\overline{RS}$ is used to measure $\overline{MN}$.



In laying off the unit along $\overline{MN}$, label P the endpoint of the first unit that falls on or beyond N and label Q the end of the preceding unit just as you did for $\overline{AB}$ on the preceding page. Using $\overline{RT}$ (which has just been determined) to aid in measuring $\overline{AB}$, we can check that $\overline{BP}$ is longer than $\overline{RT}$ and that the measure of $\overline{AB}$ is 3 , or $m(\overline{AB}) = 3$. Above, $\overline{NP}$ is shorter than $\overline{RT}$ and $m(\overline{MN}) = 4$. There is nearly always a decision to be made about whether or not to count the last unit which extends beyond the endpoint of the segment being measured. The reason for this is that it is rare indeed for the unit to fit an exact number of times from endpoint to endpoint. It is

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*Solutions for problems in the chapter are on page 312 .

well to realize now that measurement is approximate and subject to error. The "error" is the segment from the end of the segment being measured to the end of the last unit being counted. In $\overline{AB}$, the error is $\overline{BQ}$, in $\overline{MN}$, it is $\overline{NP}$. We note that the error in any measurement is always at most half the unit being used.

Let us emphasize one thing about terminology. In a phrase similar to "a line segment of 3 units" we mean "the measure of the line segment in terms of a particular unit is the number 3 ". The point here is simply to have a way of referring to the numbers involved so that they can be added, multiplied, etc. Remember that we have learned how to apply arithmetic operations only to numbers. You don't add yards any more than you add apples. If you have 3 apples and 2 apples, you have 5 apples altogether, because

$$3 + 2 = 5 .$$

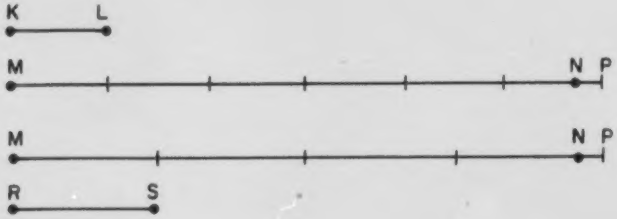
You add numbers, not yards nor apples.

As we shall see shortly, the use of different units gives rise to different measures for the same segment. Thus, if we consider $\overline{MN}$,

$m(\overline{MN}) = 6$ for the unit $\overline{KL}$ and

$m(\overline{MN}) = 4$ in terms of the unit $\overline{RS}$,

as the figure indicates.



Standard Units

Numbers of people each using their own units would have difficulty comparing their results or communicating with each other. For these reasons certain units have been agreed upon by large numbers of people and such units are called standard units.

Historically there have been many standard units used to measure line segments, such as a yard, an inch or a mile. Such a variety is a great convenience. An inch is a suitable standard unit for measuring the edge of a sheet of paper, but hardly satisfactory for finding the length of the school corridor. While a yard is a satisfactory standard for measuring the school corridor, it would not be a sensible unit for finding the distance between Chicago and Philadelphia.

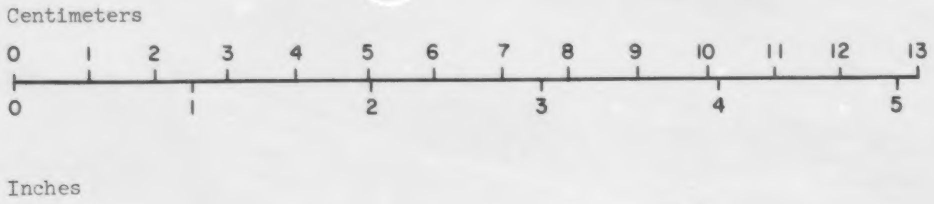
Such units of linear measure as inch, foot, yard and mile are commonly used standard units in the British-American system of measures. In the eighteenth century in France, a group of scientists developed the system of measures which is known as the metric system using a new standard unit.

In the metric system, the basic standard unit of length is the meter, which is approximately 39.37 inches or a little more than 1 yard. The metric system is in common use in all countries except those in which English is the main language spoken and is used by all scientists in the world including those in English speaking countries. Actually, the one official standard unit for linear measure even in the United States is the meter, and the correct sizes of other units such as the centimeter, inch, foot and yard are specified by law with reference to the meter.

The principal advantage of the metric system over the British-American system lies in the fact that the metric system has been designed for ease of conversion between the various metric units by exploiting the decimal system of numeration. Instead of having 12 inches to the foot, 3 feet to the yard and 1760 yards to the mile, the metric system has 10 millimeters to a centimeter, 10 centimeters to a decimeter, and 10 decimeters to a meter. This makes conversions between units very easy.

So far we have said nothing about metric units larger than the meter. The most useful of these is the kilometer, which is defined to be 1,000 meters. The kilometer is the metric unit which closely corresponds to the British-American mile. It turns out that one kilometer is a little more than six-tenths of a mile.

We have already noted that in the metric system, the meter is the unit which corresponds approximately to the yard in the British-American system. The metric unit which corresponds to the inch is the centimeter which is one-hundredth of a meter. A meter is almost 40 inches so it takes about $2\frac{1}{2}$ centimeters to make an inch or to put it another way a centimeter is about $\frac{2}{5}$ or .4 of an inch. Below are illustrated a scale of inches and a scale of centimeters so you can compare them.



Scales and Rulers

Once a standard unit such as a yard, meter or mile is agreed upon, the creation of a scale greatly simplifies measurement.

> A scale is a number line with the segment from 0 to 1 congruent to the unit being used.

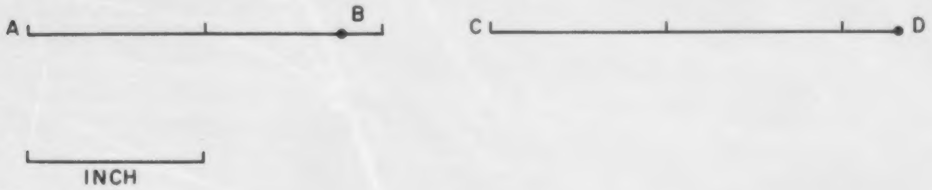
A scale can be made with a non-standard unit or with a standard unit.

> A ruler is a straight edge on which a scale using a standard unit has been marked.

If we use the inch as the unit in making a ruler, we have a measuring device designed to give us readings to the nearest inch. Most ordinary rulers are marked with the unit one-sixteenth of an inch or with the unit one millimeter.

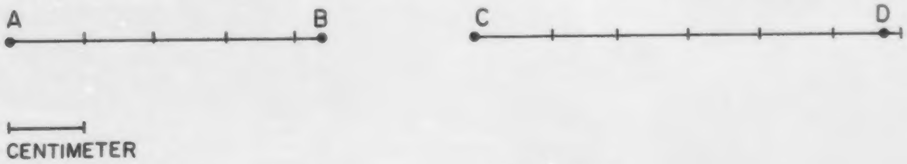
The Approximate Nature of Measure

Any measurement of the length of a segment made with a ruler is, at best, approximate. When a segment is to be measured, a scale based on a unit appropriate to the purpose of the measurement is selected. The unit is the segment with endpoints at two consecutive scale divisions of the ruler. The scale is placed on the segment with the zero-point of the scale on one endpoint of the segment. The number which corresponds to the division point of the scale nearest the other endpoint of the segment is the measure of the segment. Thus, *every measurement is made to the nearest unit*. If the inch is the unit of measure for our ruler, then we have a situation in which two line segments, apparently not the same length, may have the same measure, in terms of a specified unit.



In inches, $m(\overline{CD}) = m(\overline{AB}) = 2$

For the same two segments we may get a different measure if we use a different unit segment. It should be clear that if the unit is changed, the scale changes. Thus, if we decide to use the centimeter as our unit, the figure below shows that in centimeters $m(\overline{AB}) = 4$ and $m(\overline{CD}) = 6$. Now the measures do indicate that there is a difference in the lengths of the two segments. Notice that by using a smaller unit (the centimeter) we are able to distinguish between the lengths of two non-congruent segments which in terms of a larger unit (the inch) have the same measure. If measurements of the same segment are made in terms of different units, the error in the measurements may be different since it is at most half the unit being used. Thus, if a segment is measured in inches the error cannot be more than half an inch, while if it is measured in tenths of an inch the error cannot be more than half of a tenth of an inch. As a result, if greater precision is desired in any measurement, a smaller unit should be used.



In centimeters, $m(\overline{CD}) > m(\overline{AB})$

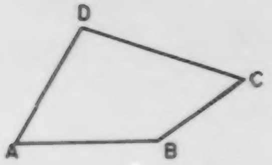
Sometimes it is more convenient to record a length of 31 inches as 2 feet 7 inches. Whenever a length is recorded using more than one unit, it is understood that the accuracy of the measure is indicated by the smallest unit named. A length of 4 yd. 2 ft. 3 in. is measured to the nearest inch. That is, it is closer to 4 yd. 2 ft. 3 in. than it is to either 4 yd. 2 ft. 2 in. or 4 yd. 2 ft. 4 in. A length of 4 yd. 2 ft. is interpreted to mean a length closer to 4 yd. 2 ft. than to 4 yd. 1 ft. or 4 yd. 3 ft. However, if this segment were measured to the nearest inch we would have to indicate this by 4 yd. 2 ft. 0 in. or 4 yd. 2 ft. (to the nearest inch). There is a very real difference in the precision of these measurements. When the measurement is made to the nearest foot, the interval within which the length may vary is one foot; when the measurement is made to the nearest inch, the interval within which the length may vary is one inch. This is because the end of the last unit counted may lie up to a half a unit on either side of the end of the segment.

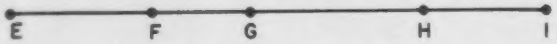
A very important property of line segments is that any line segment may be measured in terms of any given unit. This means that no matter how small the unit may be, there is a whole number n, such that if we lay off the unit n times along $\overline{AB}$ starting at A we will cover $\overline{AB}$ completely; that is, a point will be reached that is at the point B or beyond the point B on $\overrightarrow{AB}$.

The length of a line segment is a property of the line segment which we may measure in terms of different units. Theoretically, two segments have the same length if, and only if, they are congruent. We run into trouble thinking and talking about length because, in practice, measurement of length is made in terms of units and, as we saw above, two lines which are really different in length may both be said quite truly to have length 2 inches to the nearest inch.

A vivid illustration of this trouble will emerge if we think about an application of linear measurement to the calculation of the perimeter of a polygon. By definition:

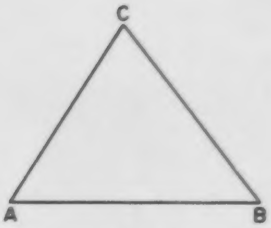
> The *perimeter* of a polygon is the length of the line segment which is the union of a set of non-overlapping line segments congruent to the sides of the polygon.

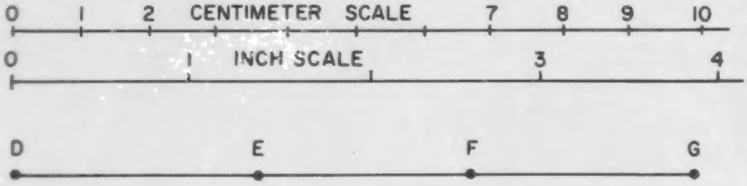




Thus the perimeter of polygon ABCD is the length of $\overline{EI}$ where $\overline{EI}$ is the union of $\overline{EF}$, $\overline{FG}$, $\overline{GH}$ and $\overline{HI}$ which are respectively congruent to $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. If we put pins at points A , B , C and D and stretch a taut thread around the polygon from A back to A , when we straighten out our thread we will have a model of a segment congruent to $\overline{EI}$.

The length of $\overline{EI}$, we know intuitively, is the sum of the lengths of the four segments when we consider length as an intrinsic property of segments. But, when we talk about lengths as measured in terms of certain units we may run into the following situation:





To the nearest centimeter $m(\overline{AB}) = m(\overline{BC}) = m(\overline{CA}) = 3$. $\overline{AB}$ is congruent to $\overline{DE}$, $\overline{BC}$ is congruent to $\overline{EF}$, $\overline{CA}$ is congruent to $\overline{FG}$ but $m(\overline{DG}) = 10$. This is because to the nearest millimeter $m(\overline{AB}) = m(\overline{BC}) = m(\overline{CA}) = 33$, and to the nearest millimeter $m(\overline{DG}) = 99$, and to the nearest centimeter this means $m(\overline{DG}) = 10$. Even if we measure our segments to the nearest inch we find $m(\overline{AB}) = m(\overline{BC}) = m(\overline{CA}) = 1$ and we would expect the measure of the perimeter to be 3 . But we find $m(\overline{DG}) = 4$. This reminds us again that the measure of a length is always, at best, an approximation and approximation errors may accumulate to cause real trouble. The best we can say is to be aware of this possibility whenever in your problems you are dealing with numbers which turn up from measurement processes. The preciseness of any measurement is related to the size of the unit selected.

Problems

3. Two children are asked to determine the length and width of a crate; one is given a ruler with units marked in feet, the other a ruler with units marked in inches. The first says the crate is 3 feet long and 2 feet wide; the second says it is 40 inches by 28 inches. Explain why they could both be right.

4. Both children are asked to find the perimeter of the crate. The first one says 10 feet, the second says 136 inches. A string is then passed around the crate, stretched out and the children are asked to measure the string to find the perimeter. This time the first one says 11 feet, the second one 137 inches. Which results are correct? Explain the discrepancy between the results.

We have indicated in this development, that length is the common property possessed by segments that are congruent in much the same way that a number is the common property of all sets that are equivalent. Corresponding to the length of a given segment, a whole number is attached which we call its measure. Note that this measure depends on the unit selected, and, as we have seen, is what one normally considers the measure to the nearest unit. Thus, length is approximated by the measure, with the approximation being closer and closer as the unit is finer and finer. This is the case for any measure whether it describes length, time, weight, or any other measurement.

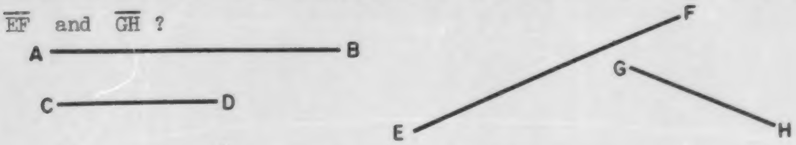
When we say that a segment has a measurement of $3\frac{1}{4}$ inches, for instance, the implication is that the unit is the quarter-inch. Thus, a "measure" of $3\frac{1}{4}$ is actually 13 , since $3\frac{1}{4}$ inches means 13 quarter-inches. When a measure is expressed as a rational number, the understanding is, therefore, that an approximation is made to the smallest unit indicated, as for example, the quarter-inch mentioned above. Starting with the concept of measure as a whole number, a meaning may now be attached to a measure given in terms of a rational number. With reference to the smaller unit, the measure is the whole number of the smaller units; with reference to the larger unit, the measure may be stated as a rational number.

On a line, a segment can always be found that would be congruent to some segment. It is then possible to choose two points on a line so that the segment determined by the two points would be congruent to the unit for a particular measure. If the two points on the line were identified as 0 and 1 , then a number line may be constructed such that the unit on the number line is congruent to the unit for the measure.

Now, suppose that the length of a given segment is to be determined. Clearly, there would be a segment on the number line from 0 to a point having a rational number as its coordinate that would approximate the given segment in length. In fact, by finding the segment on the number line with 0 as one of the endpoints (the left endpoint) that is congruent to the segment being measured, it should be possible to obtain the measure by the coordinate of the other endpoint. By this, any number that may be associated with any point on the number line as its coordinate may be assigned as the measure of a segment, and two segments are said to be of the same length if they have the same measure regardless of the unit used. Length, conceived of as the common property of congruent segments, is a slight departure from length in ordinary language usage, as for example, in stating that the length of a desk is 4 feet. The explanation of length as the common property of congruent segments more accurately emphasizes its mathematical meaning.

Exercises - Chapter 12

1. Which of the following statements is true about segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$?



a. $\overline{AB}$ is congruent to $\overline{CD}$

b. $\overline{AB}$ is shorter than $\overline{CD}$

c. $\overline{AB}$ is longer than $\overline{EF}$

d. $\overline{AB}$ is congruent to $\overline{EF}$

e. $\overline{GH}$ is shorter than $\overline{CD}$

f. $\overline{GH}$ is congruent to $\overline{CD}$

2. A dog weighs 18 pounds.

a. The unit of measure is ________.

b. The measure is ________.

c. The weight is ________.

3. A desk is 9 chalk pieces long.

a. Its measurement is ________.

b. Its measure is ________.

c. The unit of measure is ________.

4. In which of the following sentences are standard units used?

a. He is strong as an ox.

b. Put in a pinch of salt.

c. We drink a gallon of milk per day.

d. The corn is knee high.

e. I am five feet tall.

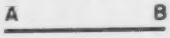
5. The measures of the sides of a triangle in inch units are 17 , 15 and 13 .

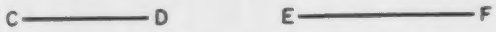
a. What are the measures of the sides if the unit is a foot?

b. What is the measure of the perimeter in inches? In feet?

c. Is there anything curious about your answer?

d. How do you explain it?

6. Use  as a unit to measure the following segments.



Is $\overline{CD}$ congruent to $\overline{EF}$? Do your answers contradict each other? Explain.

Solutions for Problems

1. a. 1 ; b. 2 ; c. 1 ; d. 2 .

2. a. 2 ; b. 3 ; c. 1 ; d. 3 . It should be noted how the measures differ.

3. 40 inches to the nearest foot is 3 feet since the error is less than $\frac{1}{2}$ foot. 28 inches to the nearest foot is 2 feet. Again the error is less than $\frac{1}{2}$ foot.

4. This problem involves the definition of perimeter of a polygon. Note that the perimeter is by definition the length of the segment which is congruent to the union of non-overlapping segments congruent to the sides. Thus the second method is the correct one for both children and the answers to the nearest unit are 11 feet and 137 inches. The first result comes from adding $3 + 2 + 3 + 2$ but each measure had an error of about 4 inches or $\frac{1}{3}$ of a foot and the accumulation of these leads to the result 10 feet which is, in fact, incorrect. The result 136 inches comes likewise because each side measured in inches had an error less than $\frac{1}{2}$ an inch but which accumulated to something near an inch. The difference between the correct results 11 feet and 137 inches is due to the fact that each child gives his answer correct to the nearest unit he is using.

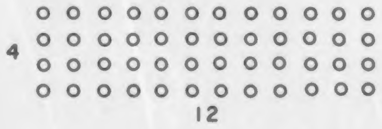
Chapter 13

MULTIPLICATION AND DIVISION TECHNIQUES

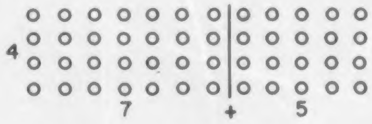
Multiplying Numbers Greater Than Ten

The ability to compute with understanding and skill when multiplying whole numbers greater than 10 depends upon several things. Among these are: knowledge of basic multiplication facts, ability to use a multiple of 10 as a factor, familiarity with our decimal place value numeration system, and ability to apply multiplication properties (commutative, associative, distributive over addition, etc.).

First let us consider the product of 4 and 12, for which we may display the array



By partitioning the array into two arrays so that each row has less than 10 members, we need to use only basic multiplication facts, the distributive property of multiplication over addition, and addition facts in order to compute the product of 4 and 12. For instance, we may partition the 4 by 12 array into a 4 by 7 array and a 4 by 5 array:

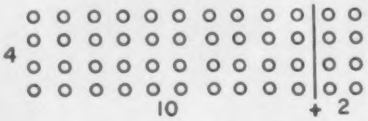


Then,
$$\begin{aligned} 4 \times 12 &= 4 \times (7 + 5) \\ &= (4 \times 7) + (4 \times 5) \\ &= 28 + 20 \\ &= 48 \end{aligned}$$

We have gone directly here from $28 + 20$ to 48 and have omitted the intervening steps:

$$\begin{aligned} &= 28 + 20 \\ &= 20 + 28 \\ &= 20 + (20 + 8) \\ &= (20 + 20) + 8 \\ &= 40 + 8 \\ &= 48 \end{aligned}$$

By choosing the numeral $7 + 5$ for 12, only basic multiplication facts from the multiplication table are needed. We could also have chosen to consider 12 as $3 + 9$, $4 + 8$ or $6 + 6$ without the necessity of going outside the table. However, since in terms of our numeration system we commonly interpret 12 as $10 + 2$, it would be more natural to partition the 4 by 12 array into two arrays in this way:



Thus,
$$\begin{aligned} 4 \times 12 &= 4 \times (10 + 2) \\ &= (4 \times 10) + (4 \times 2) \end{aligned}$$

In order to accomplish this multiplication, it is necessary to know multiplication facts for multiples of ten. This is done for the children, also.

To find the product, 4×10, we look at

$$10 + 10 + 10 + 10 = 40 .$$

Similarly, all multiples of ten are considered by adding or counting tens. Furthermore, to multiply 3 and 20 then can be thought of as:

$$\begin{aligned} 4 \times 20 &= 4 \times (10 + 10) \\ &= (4 \times 10) + (4 \times 10) \\ &= 40 + 40 \\ &= 80 \end{aligned}$$

or as

$$\begin{aligned} 4 \times 20 &= 4 \times (2 \times 10) \\ &= (4 \times 2) \times 10 \\ &= 8 \times 10 \\ &= 80 \end{aligned}$$

In the same way, multiples of tens of tens, or hundreds can be presented, and so on.

Returning to the product of 4 and 12, it can now be completed.

$$\begin{aligned} 4 \times 12 &= 4 \times (10 + 2) \\ &= (4 \times 10) + (4 \times 2) \\ &= 40 + 8 \\ &= 48 \end{aligned}$$

We often use vertical algorithms such as these to effect the same computation.

(a) $\boxed{\begin{array}{l} \quad (10 + 2) \\ \times \qquad\quad 4 \\ \hline \quad 40 + 8 = 48 \end{array}}$ or (b) $\boxed{\begin{array}{r} 10 \\ \times\ 4 \\ \hline 40 \end{array} \quad \begin{array}{r} 2 \\ \times\ 4 \\ \hline 8 \end{array} \qquad\qquad \begin{array}{r} 40 \\ +\ 8 \\ \hline 48 \end{array}}$ or

(c) $\boxed{\begin{array}{rl} 12 & \\ \times\ 4 & \\ \hline 8 & (4 \times 2) \\ 40 & (4 \times 10) \\ \hline 48 & \end{array}}$ or (d) $\boxed{\begin{array}{r} 12 \\ \times\ 4 \\ \hline 8 \\ 4 \\ \hline 48 \end{array}}$ or eventually simply (e) $\boxed{\begin{array}{r} 12 \\ \times\ 4 \\ \hline 48 \end{array}}$

As another example, consider the product of the numbers 3 and 28 :

$$\begin{aligned} 3 \times 28 &= 3 \times (20 + 8) \\ &= (3 \times 20) + (3 \times 8) \\ &= 60 + 24 \\ &= 84 \end{aligned}$$

Problem*

1. Show the multiplication of 3 and 28 in more detailed form, particularly in going from 3×20 to 60 and in going from $60 + 24$ to 84 .

We also may use one vertical algorithm or another to record our thinking when multiplying 3 and 28 :

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*Solutions for problems in this chapter are on page 325 .

(a)
$$\boxed{\begin{array}{l} \quad (20 + 8) \\ \underline{\times \qquad\quad 3} \\ \quad 60 + 24 \\ \quad 80 + 4 = 84 \end{array}}$$
or

(b)
$$\boxed{\begin{array}{rrr} 20 & 8 & 60 \\ \underline{\times 3} & \underline{\times 3} & \underline{+ 24} \\ 60 & 24 & 84 \end{array}}$$
or

(c)
$$\boxed{\begin{array}{rl} 28 & \\ \underline{\times 3} & \\ 24 & (3 \times 8) \\ \underline{60} & (3 \times 20) \\ 84 & \end{array}}$$
or

(d)
$$\boxed{\begin{array}{r} 28 \\ \underline{\times 3} \\ 24 \\ \underline{6} \\ 84 \end{array}}$$
or eventually simply

(e)
$$\boxed{\begin{array}{r} 28 \\ \underline{\times 3} \\ 84 \end{array}}$$

Now let us extend our computation to an example such as 4×236 . We shall be fairly detailed in our first illustration:

$$\begin{aligned} 4 \times 236 &= 4 \times (200 + 30 + 6) \\ &= (4 \times 200) + (4 \times 30) + (4 \times 6) \\ &= [4 \times (2 \times 100)] + [4 \times (3 \times 10)] + (4 \times 6) \\ &= [(4 \times 2) \times 100] + [(4 \times 3) \times 10] + (4 \times 6) \\ &= (8 \times 100) + (12 \times 10) + (4 \times 6) \\ &= 800 + 120 + 24 \\ &= 800 + (100 + 20) + (20 + 4) \\ &= (800 + 100) + (20 + 20) + 4 \\ &= 900 + 40 + 4 \\ &= 944 \end{aligned}$$

Problem

2. Justify each step of the procedure just illustrated for the product of 4 and 236 .

We may record our thinking in several ways using vertical algorithms:

(a)
$$\boxed{\begin{array}{l} \quad (200 + 30 + 6) \\ \underline{\times \qquad\qquad\qquad 4} \\ \quad 800 + 120 + 24 \\ \quad 900 + 40 + 4 = 944 \end{array}}$$
or

(b)
$$\boxed{\begin{array}{rrrr} 200 & 30 & 6 & 800 \\ \underline{\times 4} & \underline{\times 4} & \underline{\times 4} & 120 \\ 800 & 120 & 24 & \underline{+ 24} \\ & & & 944 \end{array}}$$
or

(c)
$$\boxed{\begin{array}{rl} 236 & \\ \underline{\times 4} & \\ 24 & (4 \times 6) \\ 120 & (4 \times 30) \\ \underline{800} & (4 \times 200) \\ 944 & \end{array}}$$
or

(d)
$$\boxed{\begin{array}{r} 236 \\ \underline{\times 4} \\ 24 \\ 12 \\ \underline{8} \\ 944 \end{array}}$$
or eventually

(e)
$$\boxed{\begin{array}{r} 236 \\ \underline{\times 4} \\ 944 \end{array}}$$

In all of these different procedures considered in this section we have seen repeatedly that use is made of the distributive property of multiplication over addition. Further extensions of multiplication to computations such as 23×45 involve even greater use of this property. However, specific consideration of these extensions is beyond the scope of this chapter.

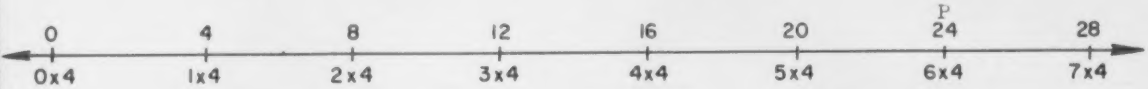
Problem

3. Use one of the vertical algorithms identified above by (a) - (e) to illustrate each of these products, a. - e. respectively. For example, use (a) as a model for a.

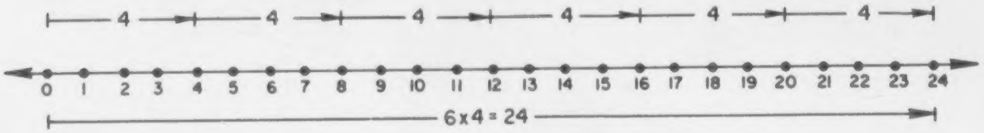
a. 3 and 23	b. 5 and 17	c. 4 and 38
d. 2 and 397	e. 6 and 130	

Division Algorithms

First let us recall that a problem such as $24 \div 4 = n$ may be interpreted to mean that we are to find the number n such that $n \times 4 = 24$. We may illustrate this in the following way, using a number line representation on which we have identified multiples of 4 :

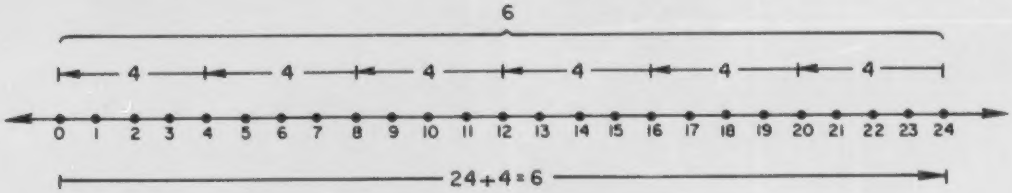


With point P we have associated 24 and also 6×4. Since the association of a number with a point is unique, we know that $6 \times 4 = 24$ and that 6 is the number n such that $n \times 4 = 24$. Let us recall what 6×4 means, using the number line. It has been interpreted in terms of repeated addition, namely $4 + 4 + 4 + 4 + 4 + 4$.



Because division is the inverse operation of multiplication and subtraction is the inverse operation of addition, it is reasonable to expect that division may be interpreted in terms of subtraction. This is indeed true.

Thus, 24 ÷ 4 can be shown on the number line as repeated subtraction.



The procedure illustrated above can be stated in terms of numbers: from 24 we subtract 4 and then continue to subtract 4 from each remainder in turn, until reaching a remainder that is less than 4 . For instance:

$$\begin{array}{r} 24 \\ -\ 4 \\ \hline 20 \end{array} \qquad \begin{array}{r} 20 \\ -\ 4 \\ \hline 16 \end{array} \qquad \begin{array}{r} 16 \\ -\ 4 \\ \hline 12 \end{array} \qquad \begin{array}{r} 12 \\ -\ 4 \\ \hline 8 \end{array} \qquad \begin{array}{r} 8 \\ -\ 4 \\ \hline 4 \end{array} \qquad \begin{array}{r} 4 \\ -\ 4 \\ \hline 0 \end{array}$$

Since there are 6 such subtractions and the resulting remainder is 0 , we know that $6 \times 4 = 24$.

Frequently we show these subtractions in a more compact form such as that shown at the right. ⟶

$$\begin{array}{r} 24 \\ -\ 4 \\ \hline 20 \\ -\ 4 \\ \hline 16 \\ -\ 4 \\ \hline 12 \\ -\ 4 \\ \hline 8 \\ -\ 4 \\ \hline 4 \\ -\ 4 \\ \hline 0 \end{array}$$

Our work might be shortened if, for instance, we subtracted multiples of 4 that are greater than 4 , such as:

$$\begin{array}{rl} 24 & \\ -\ 8 & \text{(2 fours) or } (2 \times 4) \\ \hline 16 & \\ -12 & \text{(3 fours) or } (3 \times 4) \\ \hline 4 & \\ -\ 4 & \text{(1 four) or } (1 \times 4) \\ \hline 0 & \end{array}$$

A total of 6 fours have been subtracted since

$$\begin{aligned} (2 \times 4) + (3 \times 4) + (1 \times 4) &= (2 + 3 + 1) \times 4 \\ &= 6 \times 4 \ . \end{aligned}$$

Repeated subtraction, then, provides the rationale for division algorithms. Using multiples of the divisor can be of great advantage if we are dividing larger numbers: for example, $42 \div 3 = n$.

(a)

$$\begin{array}{rrl}
 & 42 & \\
- & \underline{24} & (8 \times 3 = 24) \\
 & 18 & \\
- & \underline{15} & (5 \times 3 = 15) \\
 & 3 & \\
- & \underline{3} & \underline{(1 \times 3 = 3)} \\
 & 0 & (14 \times 3 = 42)
\end{array}$$

or simply (b)

$$\begin{array}{r|r}
3\overline{)42} & \\
\underline{24} & 8 \\
18 & \\
\underline{15} & 5 \\
3 & \\
3 & 1 \\ \hline
0 & 14
\end{array}$$

Thus, 24 ÷ 3 = 14 .

(a)

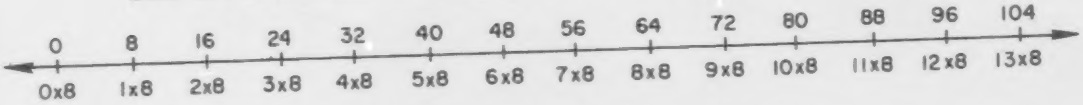
$$\begin{array}{rrl}
 & 42 & \\
- & \underline{30} & (10 \times 3 = 30) \\
 & 12 & \\
- & \underline{12} & \underline{(4 \times 3 = 12)} \\
 & 0 & (14 \times 3 = 42)
\end{array}$$

or simply (b)

$$\begin{array}{r|r}
3\overline{)42} & \\
\underline{30} & 10 \\
12 & \\
12 & 4 \\ \hline
0 & 14
\end{array}$$

As before, of course, 42 ÷ 3 = 14 , even though different multiples of 3 were used. Choosing multiples of ten may again be more natural and more simple eventually. However, children will begin with the smaller multiples and take larger jumps in accordance with their maturity.

Next let us consider an example such as 101 ÷ 8 = n .



Clearly there is no whole number n such that n × 8 = 101 , since 12 × 8 = 96 and 13 × 8 = 104 , and there is no whole number between 12 and 13 .

Let us explore the situation further in this way:

(a)

$$\begin{array}{rrl}
 & 101 & \\
- & \underline{80} & (10 \times 8 = 80) \\
 & 21 & \\
- & \underline{16} & \underline{(2 \times 8 = 16)} \\
 & 5 & (12 \times 8 = 96)
\end{array}$$

or simply (b)

$$\begin{array}{rr}
8\overline{)101} & \\
\underline{80} & 10 \\
21 & \\
16 & 2 \\ \hline
5 & 12
\end{array}$$

or eventually (c)

$$\begin{array}{r}
12 \\
8\overline{)101} \\
\underline{8} \\
21 \\
\underline{16} \\
5
\end{array}$$

Thus, although there is no whole number n such that n × 8 = 101 , we have determined that 101 = 96 + 5 or 101 = (12 × 8) + 5 . However, we are not permitted to write something such as $101 \div 8 = 12 \text{ r } 5$, since "12 r 5" is not a name for a number.

In general, if a is any whole number and b is any counting number, we may associate with $a \div b$ or $\frac{a}{b}$ the sentence

$$a = (n \times b) + r$$

commonly written in the form

$$\begin{array}{r} n\,\frac{r}{b} \\ b\overline{)a} \end{array}$$

for which n is a unique whole number such that $(n \times b) \leq a$ and $r < b$. For example, $20 \div 3$ can be associated with

$$20 = (6 \times 3) + 2 \quad \text{or} \quad \begin{array}{r} 6\frac{2}{3} \\ 3\overline{)20} \end{array}$$

where $a = 20$, $b = 3$, $n = 6$ and $r = 2$. In more detail, the common algorithm would appear:

$$\begin{array}{r} 6\,\frac{2}{3} \\ 3\overline{)20} \\ \underline{18} \\ 2 \end{array}$$

Thus, $6 \times 3 = 18$ is to be subtracted from 20 to find the remainder. In order to subtract, then, 18 must be less than or equal to 20 . If the remainder, 2 , had been greater than or equal to 3 , we could have found a larger multiple of 3 to subtract from 20 .

The condition that $r < b$ has a further implication. It is certainly true that $20 \div 3$ can be associated with this equation:

$$20 = (1 \times 3) + 17$$

from which it can be stated that $20 \div 3$ is 1 with a remainder of 17. Similarly,

$$\begin{aligned} 20 &= (2 \times 3) + 14 \\ 20 &= (3 \times 3) + 11 \\ 20 &= (4 \times 3) + 8 \\ 20 &= (5 \times 3) + 5 \\ 20 &= (6 \times 3) + 2 \end{aligned}$$

are all valid equations associated with $20 \div 3$. It is generally understood, however, that when we wish to know what 20 divided by 3 is, we want the quotient expressed as the largest possible whole number plus a nonnegative remainder. (Note that there is always a remainder. When b is a factor of a, it happens to be 0 .) Thus by restricting the remainder, r, to be less than the divisor, b, we assure that n will be the largest whole number of times b is contained in a, and so we only associate with $20 \div 3$ the equation we want:

$$20 = (6 \times 3) + 2 .$$

Now let us use division algorithms to find n and r for this expression: $250 \div 7$ or $\frac{250}{7}$.

(a)

```
   250
 - 140    (20 × 7)
   110
 -  70    (10 × 7)
    40
 -  28    ( 4 × 7)
    12
 -   7    ( 1 × 7)
     5    (35 × 7)
```

or (b)

```
7)250 |
  140 |  20
  110 |
   70 |  10
   40 |
   28 |   4
   12 |
    7 |   1
    5 |  35
```

or, using larger multiples:

```
7)250 |
  210 |  30
   40 |
   35 |   5
    5 |  35
```

or eventually (c)

```
     35
  7)250
    21
     40
     35
      5
```

Thus, for $a = 250$ and $b = 7$, we see that $n = 35$ and $r = 5$.

We therefore may associate with $250 \div 7$ or $\frac{250}{7}$ the sentence

$$250 = (35 \times 7) + 5 .$$

Problems

4. For each of the following write an equation of the form $a = (n \times b) + r$, such that $(n \times b) \leq a$ and $r < b$.

a. $38 \div 5$	b. $79 \div 3$	c. $112 \div 4$
d. $\frac{57}{6}$	e. $\frac{83}{3}$	f. $\frac{106}{2}$

5. Rewrite the general equation for the special case where $r = 0$.

Consider the example $74 \div 3 = n$, or $\frac{74}{3} = n$:

$$\begin{array}{r|r} 3\overline{)74} & \\ 60 & 20 \\ \hline 14 & \\ 12 & 4 \\ \hline 2 & 24 \end{array}$$

This algorithm provides us with a great deal of information.

First, since the remainder is not zero, we know that there is no whole number n such that $3 \times n = 74$. That is, 3 is not a factor of 74 .

Second, the algorithm gives us the information we need to replace n and r in the equation $74 = (n \times 3) + r$ so that $(n \times 3) \leq 74$ and $r < 3$. We now may write

$$74 = (24 \times 3) + 2 .$$

Third, although there is no whole number n such that $3 \times n = 74$, there very definitely is a rational number n such that $3 \times n = 74$. One name for that rational number is $\frac{74}{3}$, since $3 \times \frac{74}{3} = 74$. The algorithm gives us the information needed to name this rational number in a different way, in mixed form. From our knowledge of rational numbers we know that 2 (the remainder) is $\frac{2}{3}$ of 3 (the divisor); that is, $2 = \frac{2}{3} \times 3$. We then may assert that

$$74 \div 3 = 24 + \frac{2}{3} = 24\frac{2}{3} \quad \text{or} \quad \frac{74}{3} = 24 + \frac{2}{3} = 24\frac{2}{3} .$$

Thus, we know that

$$3 \times (24 + \frac{2}{3}) = 3 \times 24\frac{2}{3} = 74 .$$

Divisions with larger numbers follow the same ideas we have developed but are beyond the scope of this chapter.

Problem

6. For each exercise of Problem 4, express the quotient as a rational number in mixed form or as a whole number.

Summary

In the development of multiplication algorithms we used extensively the distributive property of multiplication over addition, coupled with the renaming of a factor in accord with our decimal place value numeration scheme. For instance, in order to effect the product of 4 and 23, we renamed 23 as $(20 + 3)$ and then applied the distributive property:

$$4 \times (20 + 3) = (4 \times 20) + (4 \times 3) .$$

In the development of division algorithms we utilized a process of "repeated subtraction" in which we successively subtracted multiples of the divisor. We saw that the greater the size of the multiples used, the more efficient is the algorithm.

The division algorithm gives the information necessary to associate with $a \div b$ or $\frac{a}{b}$ (where $\underline{a}$ is any whole number and b is any counting number) either of two things:

1. an equation of the form $a = (n \times b) + r$, where $(n \times b) \leq a$ and $r < b$,
2. a rational number in mixed form whenever $a > b$ and b is not a factor of $\underline{a}$.

A special case of both 1. and 2. arises when $r = 0$; that is, when b is a factor of $\underline{a}$.

Applications to Teaching

It is important that algorithms are developed from the standpoint of being written records of thinking patterns used when computing. Thus, we can expect that children's algorithms will change with the passing of time. At first the multiplication and division algorithms may be more lengthy and less efficient than at a later stage of work. We should allow children to use those algorithms that are most helpful and sensible to *them*. We may encourage them to shorten algorithms over a period of time, but children should not be forced to use more efficient algorithms prematurely.

Exercises - Chapter 13

1. Use several different algorithms to compute each of these:

   a. 7 × 34 c. 9 × 28

   b. 6 × 48 d. 8 × 54

2. Associate two things with each of the following: an equation of the form $a = (n \times b) + r$ where $(n \times b) \leq a$ and $r < b$; and a rational number in mixed form (or a whole number if b is a factor of a).

   a. 38 ÷ 6 c. 125 ÷ 8

   b. 99 ÷ 4 d. 84 ÷ 3

3. a. Using the common division algorithm, find the quotient 342 ÷ 7 .

   b. Relate this algorithm to the more primitive algorithms used by the children when they are first introduced to division.

4. In $a = (n \times b) + r$, explain why $n \times b \leq a$ and $r < b$.

Solutions for Problems

1. $3 \times 28 = 3 \times (20 + 8)$
$= (3 \times 20) + (3 \times 8)$
$= (3 \times 2 \times 10) + (3 \times 8)$
$= (6 \times 10) + (3 \times 8)$
$= 60 + 24$
$= (60 + 20) + 4$
$= 80 + 4$
$= 84$

2. $4 \times 236 = 4 \times (200 + 30 + 6)$ Renaming 236
$= (4 \times 200) + (4 \times 30) + (4 \times 6)$ Distributive property of multiplication over addition
$= [4 \times (2 \times 100)] + [4 \times (3 \times 10)] + (4 \times 6)$ Renaming
$= [(4 \times 2) \times 100] + [(4 \times 3) \times 10] + (4 \times 6)$ Associative property of multiplication
$= (8 \times 100) + (12 \times 10) + (4 \times 6)$ Multiplying
$= 800 + 120 + 24$ Multiplying
$= 800 + (100 + 20) + (20 + 4)$ Renaming
$= (800 + 100) + (20 + 20) + 4$ Associative property of addition
$= 900 + 40 + 4$ Adding
$= 944$ Adding

3. a. $\begin{array}{r} (20 + 3) \\ \times \quad\quad 3 \\ \hline 60 + 9 \end{array} = 69$

b. $\begin{array}{r} 10 \\ \times\ 5 \\ \hline 50 \end{array} \quad \begin{array}{r} 7 \\ \times\ 5 \\ \hline 35 \end{array} \quad \begin{array}{r} 50 \\ +\ 35 \\ \hline 85 \end{array}$

c. $\begin{array}{r} 38 \\ \times\ 4 \\ \hline 32 \\ 120 \\ \hline 152 \end{array}$

d. $\begin{array}{r} 397 \\ \times\quad 2 \\ \hline 14 \\ 18 \\ 6 \\ \hline 794 \end{array}$

e. $\begin{array}{r} 130 \\ \times\ \ 6 \\ \hline 780 \end{array}$

4. a. $38 = (7 \times 5) + 3$ b. $79 = (26 \times 3) + 1$ c. $112 = (28 \times 4) + 0$
d. $57 = (9 \times 6) + 3$ e. $83 = (27 \times 3) + 2$ f. $106 = (53 \times 2) + 0$

5. $a = n \times b$

6. a. $7\frac{3}{5}$ b. $26\frac{1}{3}$ c. 28 d. $9\frac{3}{6} = 9\frac{1}{2}$

e. $27\frac{2}{3}$ f. 53

Chapter 14

STRUCTURE

The Counting Numbers

In our development, we have started with sets as pre-number concepts and obtained from them the set of counting (natural) numbers. Although we did not consider the properties of the counting numbers (we considered properties of whole numbers), if we had examined the counting numbers in this light, we would have discovered closure under addition and multiplication. In fact, all of the properties listed below hold for the set of counting numbers:

> the set is closed under addition and multiplication;
> the elements are commutative under addition and multiplication;
> the elements are associative under addition and multiplication;
> there is an identity element for multiplication;
> multiplication is distributive over addition.

The statement for the closure property under addition is: if $\underline{a}$ and b are counting numbers, then $a + b$ is a counting number. This may also be stated:

> if $\underline{a}$ and b are counting numbers, and
> $a + b = c$, then c is a counting number.

Thus, if $\underline{a}$ is 3 and b is 5 , then c is $3 + 5$, or 8 . A related question is: if $\underline{a}$ is 3 and c is 8 , is there a counting number x such that $a + x = c$? In terms of open sentences, we are then looking for the solution for

$$3 + x = 8 .$$

In this case, 5 is the solution of the equation. If we ask whether there is a counting number b such that $3 + b = 8$, we are posing the question: Is $3 + b = 8$ solvable in the set of counting numbers?

The Whole Numbers

In our study, we have found that $3 + 0 = 3$; furthermore, 0 is the only solution for $3 + x = 3$. However, 0 is not a counting number.

Clearly then, $3 + x = 3$ is not solvable in the set of counting numbers. Nor are $5 + x = 5$, $6 + x = 6$, $2 + x = 2$, and so on. In fact, for any counting number $\underline{a}$, 0 is the only solution for

$$a + x = a ,$$

and hence, $a + x = a$ has no solution in the set of counting numbers.

By adjoining 0 to the set of counting numbers, we obtain an extension from the counting numbers to the whole numbers. That is,

$$\text{if } Z = \{0\} \text{ and } N = \{1, 2, 3, 4, 5, \ldots\} ,$$
$$\text{then } Z \cup N = \{0, 1, 2, 3, 4, 5, \ldots\} = W .$$

Within the set of whole numbers, then, the equation $a + x = a$ has the solution $x = 0$. All the properties that we have for the set of counting numbers hold equally for the set of whole numbers. By the inclusion of 0 in the set of whole numbers some new properties are gained:

there is an identity element for addition;
the product of 0 and any whole number is 0 .

The Integers

Even adjoining 0 to the set of counting numbers is not enough to completely solve the equation, $a + x = c$. If $c < a$, this equation is not solvable in the set of whole numbers. For example, there is no whole number x such that $5 + x = 3$. Negative numbers are introduced in the first grade, but only in a limited way in relation to the number line, for example, as associated with the scale on a thermometer. Later on, when negative numbers are explored in greater detail, the opposites of the counting numbers, namely, $\{\ldots , {}^-4 , {}^-3 , {}^-2 , {}^-1\}$, may be adjoined to the whole numbers. Thus, we get the set of integers

$$I = \{\ldots , {}^-4 , {}^-3 , {}^-2 , {}^-1 , 0 , 1 , 2 , 3 , \ldots\} .$$

Then, the equation $a + x = c$ will be solvable in the set of integers for numbers $\underline{a}$ and c in this set. By this extension, we will find that all the properties that we have identified for the whole numbers still hold for the integers. Moreover, we have an additional property which derives from the solvability of $a + x = 0$ for any integer $\underline{a}$. The solution for this equation is called the inverse of $\underline{a}$. The property may be stated:

> for each integer $\underline{a}$, there is an inverse,
> ${}^{-}a$ such that $a + {}^{-}a = 0$.

By the commutative property, we can see that $\underline{a}$ and ${}^{-}a$ are inverses of each other. For example, $3 + {}^{-}3 = 0$ and ${}^{-}3 + 3 = 0$; so 3 and ${}^{-}3$ are inverses of each other.

Historically, there was only need of the counting numbers for the primitive man; his possessions and all his reckoning were adequately accounted for by these numbers. The concept of zero as a number did not emerge until quite late in civilization. With sophistication, we may interpret the concept from a different point of view. Zero might be considered to be the solution for $a + x = a$ for whatever number $\underline{a}$; in this way, a number called zero is "postulated" as the solution. Similarly, ${}^{-}a$ may be postulated as the solution for $a + x = 0$.

The Rational Numbers

We may next consider the solvability of equations of the form $a \times x = c$ for integers $\underline{a}$ and c . Evidently, for certain numbers such as $a = 2$ and $c = 6$, the equation $a \times x = c$ is solvable in integers. The solution for $2 \times x = 6$ is 3 . However, equations such as

$$6 \times x = 2 ,$$

are not solvable in the set of integers. This leads to the set of all rational numbers: numbers represented by $\frac{m}{n}$ where m and n are integers and $n \neq 0$. The solution for $6 \times x = 2$ is then considered to be $\frac{2}{6}$ just as the solution for $2 \times x = 6$ is considered to be $\frac{6}{2}$. As we have indicated in the preceding section regarding the postulation of zero and ${}^{-}a$, the number $\frac{m}{n}$ may also be postulated as the solution for $n \times x = m$.

By representation of such numbers on the number line, we identified, for example, the numbers named as

$$\frac{3}{1},\ \frac{6}{2},\ \frac{9}{3},\ \ldots,\ \frac{m \times k}{n \times k},\ \ldots,\ \text{for } k \neq 0$$

to be the same number. Thus,

> if a and b are nonnegative integers such that $b \times k \neq 0$, then all numbers that can be represented by $\frac{a \times k}{b \times k}$ are identified with $\frac{a}{b}$ and all numbers that can be represented by $\frac{{}^{-}a \times k}{b \times k}$ are identified with $\frac{{}^{-}a}{b}$, where a and b do not have any common factor other than 1 (unless $a = 0$).

In this way, $\frac{1}{3}$, $\frac{2}{6}$, $\frac{3}{9}$, ... are considered to be in the same "equivalence" class; $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, ... in another equivalence class; $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, ... in still another class; and so on. Corresponding to the equivalence of $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, ... is the equivalence of the statements

$$3 \times x = 2, \quad 6 \times x = 4, \quad 9 \times x = 6, \quad \ldots .$$

So, instead of defining the equivalence classes via the number line, the concept also can be approached via equivalent statements. Either way, $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, ... would be classified together. Our approach by the number line is the more intuitive approach in accord with the presentation to the students.

There is another kind of identification that we might interpret by the number line. It is that the rational numbers $\frac{m}{1}$, $\frac{m \times 2}{1 \times 2}$, $\frac{m \times 3}{1 \times 3}$, ..., may be identified with the integer m, if m is an integer. From this viewpoint, the set of rational numbers may be regarded as an extension of the set of integers. We can observe that in the set of rationals, all the properties that we have identified that hold for the integers still hold. Furthermore, another property is gained -- one that parallels the property on inverses under addition:

> for each rational number $\frac{m}{n}$ that is different from 0, there is an inverse $\frac{p}{q}$ such that $\frac{m}{n} \times \frac{p}{q} = \frac{1}{1}$ (with the identification, $\frac{1}{1} = 1$).

For example, $\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = \frac{6}{6} = \frac{1}{1}$.

With extension on top of extension, we see an emerging structure of the numbers as characterized by the properties. Each set of numbers, together with the operations and the properties, form what is called a number system. For the rational number system, the properties may be listed as follows:

the set is closed under addition and multiplication, for example, $\frac{1}{2} + \frac{5}{3}$ is a rational number;

the elements are commutative under addition and multiplication, for example, $\frac{1}{2} + \frac{5}{3} = \frac{5}{3} + \frac{1}{2}$;

the elements are associative under addition and multiplication, for example, $(\frac{1}{2} + \frac{5}{3}) + \frac{3}{4} = \frac{1}{2} + (\frac{5}{3} + \frac{3}{4})$

there is an identity element for addition, for example, $\frac{1}{2} + 0 = \frac{1}{2}$;

there is an identity element for multiplication, for example, $\frac{3}{4} \times 1 = \frac{3}{4}$;

for each rational number, there is an inverse under addition, for example, $\frac{2}{3} + {}^{-}(\frac{2}{3}) = 0$;

for each rational number different from 0 , there is an inverse under multiplication, for example, $\frac{5}{6} \times \frac{6}{5} = 1$;

multiplication is distributive over addition, for example, $\frac{1}{2} \times (\frac{2}{3} + \frac{5}{7}) = (\frac{1}{2} \times \frac{2}{3}) + (\frac{1}{2} \times \frac{5}{7})$.

Besides these, there are properties which we can elicit from the above, such as

the product of 0 and any rational number is 0 ; for example, $0 \times \frac{9}{7} = 0$.

Other Extensions

Other extensions will be made beyond the set of rational numbers, but these will not be carried out in the first six grades. The rational numbers were associated with points on the number line. As the rational numbers have the property of being dense (between any two rational numbers are infinitely many rational numbers), it appears that every point on the number line represents a rational number. However, there are numbers such as π, $\sqrt{2}$, $\sqrt[3]{7}$, and so on, that are coordinates of points on the number line but are not rational numbers.

The next extension brings us the set of all numbers that may be represented on the number line. These are the real numbers. Beyond this extension are the complex numbers, whose representations occupy the entire coordinate plane (that is, just the number line is not sufficient for their respresentations) and the hypercomplex numbers. With each number system is associated a structure given by its properties.

We have pointed to the property or properties gained with each extension. However, although we shall not show how here, we should mention that it is not always the case that properties are gained. The extension from the complex numbers to a hypercomplex system may result in the loss of the commutative property; a further extension may result in the loss of both the commutative and associative properties.

There are other losses of properties that occur in the extensions which have not been mentioned but which we will note very briefly now. When the set of whole numbers is extended to the set of integers, we lose the property that there is a number which we can call a first (or smallest) number. Extending to the rationals, we lose the property that each number has a number which we call the next number (or successor). That is, the integers can be visualized as "isolated" (discrete) points on the number line, whereas the rationals are visualized as being densely packed. It can be shown that the rational numbers may be put into 1-1 correspondence with the counting numbers, whereas a 1-1 correspondence cannot be made with the real numbers (we say that we lose the property of countability in the extension). The extension from the real numbers to the complex numbers results in loss of the property of order: between two complex numbers, there is no "order relation" such as "<" or ">" that determines which of the two numbers precedes the other.

While we have losses with the extensions mentioned, the gains apparently far outweigh the losses, considering the many, many new problems that can be solved with each extension. An important aspect in the study of algebraic extensions consists of determining properties that hold in each extension. In turn, the study may orient itself to investigating what extensions may be determined that would retain certain properties (such as associativity, etc.), and this is indeed a program in the study of algebra.

An appropriate observation to make at this time is that in presenting mathematics as a structured discipline, the student is guided through the extensions of the number systems. Thus, with the student's maturity, his knowledge of systems of numbers is simultaneously broadened and deepened.

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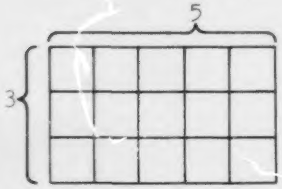
ANSWERS TO EXERCISES

Chapter 6

1. $1000 \times 3 = \overbrace{3 + 3 + 3 + 3 + \cdots + 3}^{1000 \text{ addends}} = 3000$. This expresses 1000×3 .
by the commutative property of multiplication, $1000 \times 3 = 3 \times 1000$;
and $3 \times 1000 = 1000 + 1000 + 1000 = 3000$.

2. a. $4 \times 5 = 20$; b. $3 \times 2 = 6$;
c. $2 \times 4 = 8$; d. $3 \times 3 = 9$

3.



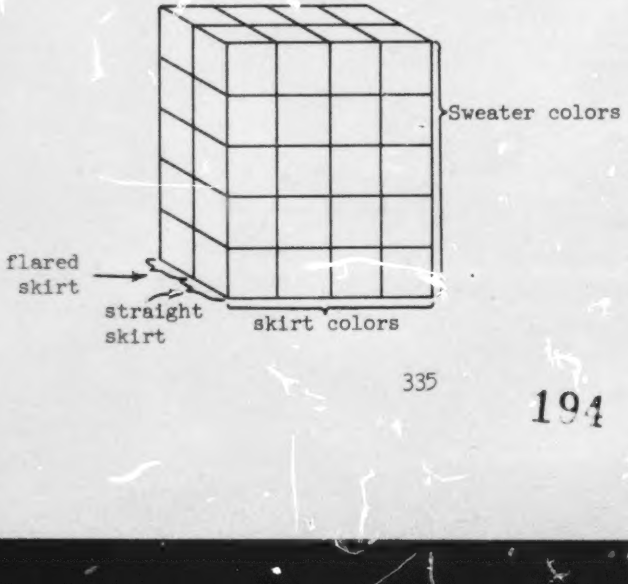
4.

	red	orange	yellow	green	blue
red	red red	orange red	yellow red	green red	blue red
yellow	red yellow	orange yellow	yellow yellow	green yellow	blue yellow
blue	red blue	orange blue	yellow blue	green blue	blue blue

15 possible results.

If the car must be two-toned, there are only 12 choices.

5.



$2 \times 4 \times 5 = 40$.
40 different ensembles.

6. a. $n = 32$; $p = 12$; $q = 20$
b. yes ; c. yes

7. The star pattern does not give 5 disjoint sets with 4 members in each set.

8. $20 \times (28 + 11 + 11) = (20 \times 28) + (20 \times 11) + (20 \times 11)$
$= 560 + 220 + 220$
$= 560 + 440$
$= 1000$

or $20 \times (28 + 11 + 11) = 20 \times (39 + 11)$
$= 20 \times (50)$
$= 1000$

9. Associative property of addition.

10. a. $(5 \times 2) \times (4 \times 3) \times 1 = 10 \times 12 = 120$
b. $(125 \times 8) \times (7 \times 3) = 1000 \times 21 = 21{,}000$
c. $(250 \times 4) \times (14 \times 2) = 1000 \times 28 = 28{,}000$

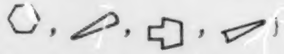
11. Commutative property under multiplication.

12. a. $3 \times (4 + \underline{3}) = (3 \times 4) + (3 \times 3)$
b. $2 \times (\underline{4} + 5) = (2 \times 4) + (\underline{2} \times 5)$
c. $13 \times (16 + 4) = (13 \times \underline{16}) + (13 \times \underline{4})$
d. $(2 \times 7) + (3 \times \underline{7}) = (\underline{2} \times \underline{3}) \times 7$

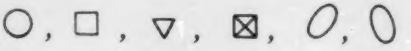
13. a. $B = \{\ \}$. Then $A \cup B = A \cup \{\ \} = A$
and $N(A \cup B) = N(A) + N(B) = N(A) + 0 = N(A)$.

b. Although the empty set is a subset of every set, it has no members in common with any other set. Therefore, any set A and the empty set are disjoint.

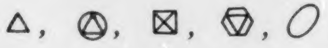
Chapter 7

1. C = {  }

Joining C to B yields B ∪ C = A .

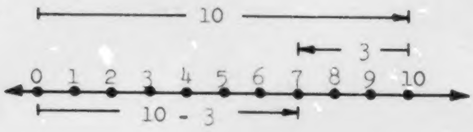
2. A ~ B = {  }

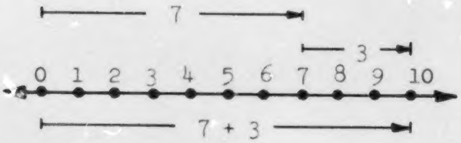
3. 6

4. B = {  }

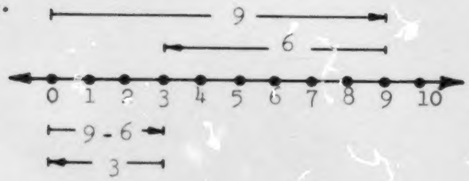
N(B) = 5

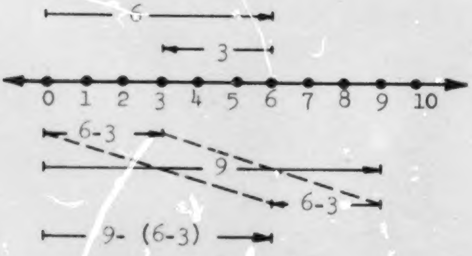
5.


10
3
0 1 2 3 4 5 6 7 8 9 10
10 - 3


7
3
0 1 2 3 4 5 6 7 8 9 10
7 + 3

6.


9
6
0 1 2 3 4 5 6 7 8 9 10
9 - 6
3


6
3
0 1 2 3 4 5 6 7 8 9 10
6-3
9
6-3
9- (6-3)

7. Subtracting 7 from the sum.

Adding 8 to the difference.

8. Let $A = \{ \bigcirc , \triangle , \square \}$ and $B = \{a , b , c\}$.
Then $A \cup B = \{ \bigcirc , \triangle , \square , a , b , c\}$
and $(A \cup B) \sim B = \{ \bigcirc , \triangle , \square \} = A$.

If A and B are not disjoint, the sets $(A \cup B) \sim B$ and A are not equal. See example.

$A = \{a , b , c , d , e\}$; $B = \{a , d , g , j\}$.
$A \cup B = \{a , b , c , d , e , g , j\}$.
$(A \cup B) \sim B = \{b , c , e\}$, which is a new set.

9. a. $n = 20 \div 5$; $n = 4$
b. $p = 28 \div 4$; $p = 7$
c. $n = 6 \div 1$; $n = 6$
d. $n = 72 \div 9$; $n = 8$
e. $n = 64 \div 8$; $n = 8$

f. No division sentence can be written. Division by 0 is undefined. $q \times 0 = 0$ is true for any number q .

10. a. Rectangular array with 7 rows and 6 columns.
b. Disjoint subsets, six with seven members each.

Either interpretation is equally valid. There may be slight preference in thinking of disjoint subsets in b , since subsets of seven members each are specified in the packaging.

11. The number 59 is a prime number, so no rectangular arrays can be formed other than one with a single row or a single column. Sixty members allows many rectangular formations since its factors are 1 , 2 , 3 , 4 , 5 , 6 , 10 , 12 , 15 , 20 , 30 , 60 .

12. No. $15 \div 5 \neq 5 \div 15$. In fact, $5 \div 15$ has no meaning in the set of whole numbers.

13. a. 2×6 or 3×4
    b. 2×18 ; 3×12 ; 4×9 ; or 6×6
    c. Prime
    d. Prime
    e. 2×4
    f. Prime
    g. 5×7
    h. Prime
    i. 3×13
    j. 2×21 ; 3×14 ; or 6×7
    k. 2×3
    l. Prime
    m. 2×41
    n. 5×19

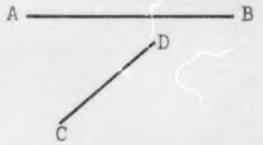
Chapter 8

1. a. $>$
   b. $<$
   c. $<$
   d. $>$
   e. $=$
   f. $>$

2. a. 14_{eight}
   b. 6_{seven}
   c. 62_{nine}
   d. 5_{six}

Chapter 9

1. Since segments have two endpoints, it is quite possible for them not to intersect and yet not lie in parallel lines. $\overline{AB}$ and $\overline{CD}$ illustrate two segments which satisfy the conditions of lying in the same plane and not intersecting; however, they are not parallel.

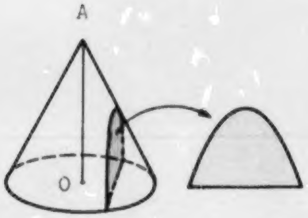


2. The line; a point; { } .

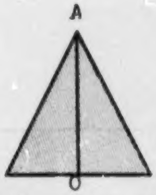
3. Model construction.

4. a. 6 ; b. 8 ; c. $2 \times n$

5.



or



if the plane contains the line of center

6. a. ; b.
   c. does not have to be. When the quadrilateral is not convex, the pyramid is not.

7. $\overline{XZ}$, $\overline{ZY}$, $\overline{XY}$

8. $\overleftrightarrow{AB}$ contains the point A ; A is in the angle, not in its interior.

1. a.

$$\begin{array}{rcrcr} 2 \text{ hundreds} & + & 4 \text{ tens} & + & 6 \text{ ones} \\ 1 \text{ hundred} & + & 3 \text{ tens} & + & 9 \text{ ones} \\ \hline 3 \text{ hundreds} & + & 7 \text{ tens} & + & 15 \text{ ones} \\ 3 \text{ hundreds} & + & 8 \text{ tens} & + & 5 \text{ ones} = 385 \end{array}$$

or

$$\begin{array}{rcrcr} 200 & + & 40 & + & 6 \\ 100 & + & 30 & + & 9 \\ \hline 300 & + & 70 & + & 15 \\ 300 & + & 80 & + & 5 = 385 \end{array}$$

or

$$\begin{array}{r} 246 \\ 139 \\ \hline 15 \\ 70 \\ 300 \\ \hline 385 \end{array}$$

b.

$$\begin{array}{rcrcr} 7 \text{ hundreds} & + & 7 \text{ tens} & + & 7 \text{ ones} \\ 9 \text{ hundreds} & + & 6 \text{ tens} & + & 4 \text{ ones} \\ \hline 16 \text{ hundreds} & + & 13 \text{ tens} & + & 11 \text{ ones} \\ 17 \text{ hundreds} & + & 4 \text{ tens} & + & 1 \text{ ones} = 1741 \end{array}$$

or

$$\begin{array}{rcrcr} 700 & + & 70 & + & 7 \\ 900 & + & 60 & + & 4 \\ \hline 1600 & + & 130 & + & 11 \\ 1700 & + & 40 & + & 1 = 1741 \end{array}$$

or

$$\begin{array}{r} 777 \\ 964 \\ \hline 11 \\ 130 \\ 1600 \\ \hline 1741 \end{array}$$

c.

$$\begin{array}{rcrcr} 4 \text{ hundreds} & + & 8 \text{ tens} & + & 6 \text{ ones} \\ 7 \text{ hundreds} & + & 6 \text{ tens} & + & 6 \text{ ones} \\ \hline 11 \text{ hundreds} & + & 14 \text{ tens} & + & 12 \text{ ones} \\ 12 \text{ hundreds} & + & 5 \text{ tens} & + & 2 \text{ ones} = 1252 \end{array}$$

or

$$\begin{array}{rcrcr} 400 & + & 80 & + & 6 \\ 700 & + & 60 & + & 6 \\ \hline 1100 & + & 140 & + & 12 = 1252 \end{array}$$

or

$$\begin{array}{r} 486 \\ 766 \\ \hline 12 \\ 140 \\ 1100 \\ \hline 1252 \end{array}$$

d.

$$\begin{array}{rcrcr} 7 \text{ hundreds} & + & 7 \text{ tens} & + & 4 \text{ ones} \\ 9 \text{ hundreds} & + & 2 \text{ tens} & + & 6 \text{ ones} \\ \hline 16 \text{ hundreds} & + & 9 \text{ tens} & + & 10 \text{ ones} \\ 17 \text{ hundreds} & + & 0 \text{ tens} & + & 0 \text{ ones} = 1700 \end{array}$$

or

$$\begin{array}{rcrcr} 700 & + & 70 & + & 4 \\ 900 & + & 20 & + & 6 \\ \hline 1600 & + & 90 & + & 10 \\ 1700 & + & 0 & + & 0 = 1700 \end{array}$$

or

$$\begin{array}{r} 774 \\ 926 \\ \hline 10 \\ 90 \\ 1600 \\ \hline 1700 \end{array}$$

2. a.

$$\begin{array}{lcl}
\text{7 hundreds + 6 tens + 4 ones} & = & \text{6 hundreds + 15 tens + 14 ones} \\
\underline{\text{1 hundred + 9 tens + 9 ones}} & = & \underline{\text{1 hundred + 9 tens + 9 ones}} \\
 & & \text{5 hundreds + 6 tens + 5 ones} = 565
\end{array}$$

or

$$\begin{array}{lcl}
700 + 60 + 4 & = & 600 + 150 + 14 \\
\underline{100 + 90 + 9} & = & \underline{100 + 90 + 9} \\
 & & 500 + 60 + 5 = 565
\end{array}$$

b.

$$\begin{array}{lcl}
\text{4 hundreds + 0 tens + 2 ones} & = & \text{3 hundreds + 9 tens + 12 ones} \\
\underline{\text{1 hundred + 3 tens + 8 ones}} & = & \underline{\text{1 hundred + 3 tens + 8 ones}} \\
 & & \text{2 hundreds + 6 tens + 4 ones} = 264
\end{array}$$

or

$$\begin{array}{lcl}
400 + 0 + 2 & = & 300 + 90 + 12 \\
\underline{100 + 30 + 8} & = & \underline{100 + 30 + 8} \\
 & & 200 + 60 + 4 = 264
\end{array}$$

c.

$$\begin{array}{lcl}
\text{7 hundreds + 1 ten + 0 ones} & = & \text{6 hundreds + 10 tens + 10 ones} \\
\underline{\text{2 hundreds + 8 tens + 7 ones}} & = & \underline{\text{2 hundreds + 8 tens + 7 ones}} \\
 & & \text{4 hundreds + 2 tens + 3 ones} = 423
\end{array}$$

or

$$\begin{array}{lcl}
700 + 10 + 0 & = & 600 + 100 + 10 \\
\underline{200 + 80 + 7} & = & \underline{200 + 80 + 7} \\
 & & 400 + 20 + 3 = 423
\end{array}$$

d.

$$\begin{array}{lcl}
\text{8 hundreds + 0 tens + 0 ones} & = & \text{7 hundreds + 9 tens + 10 ones} \\
\underline{\text{3 hundreds + 9 tens + 6 ones}} & = & \underline{\text{3 hundreds + 9 tens + 6 ones}} \\
 & & \text{4 hundreds + 0 tens + 4 ones} = 404
\end{array}$$

or

$$\begin{array}{lcl}
800 + 0 + 0 & = & 700 + 90 + 10 \\
\underline{300 + 90 + 6} & = & \underline{300 + 90 + 6} \\
 & & 400 + 0 + 4 = 404
\end{array}$$

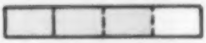
3.

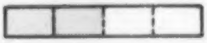
$$\begin{aligned}
774 + 926 &= (700 + 70 + 4) + (900 + 20 + 6) \\
&= (700 + 900) + (70 + 20) + (4 + 6) \\
&= 1600 + 90 + 10 \\
&= 1600 + 100 \\
&= 1700
\end{aligned}$$

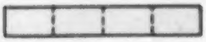
4.

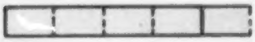
$$\begin{aligned}
800 - 396 &= 800 - (300 + 90 + 6) \\
&= (700 + 90 + 10) - (300 + 90 + 6) \\
&= (700 - 300) + (90 - 90) + (10 - 6) \\
&= 400 + 4 \\
&= 404
\end{aligned}$$

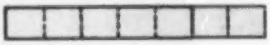
Chapter 11

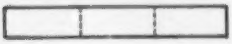
1. a. 

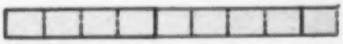
b. 

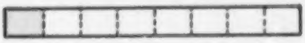
c. 

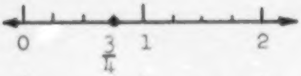
d. 

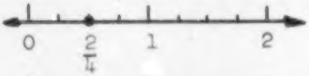
e. 

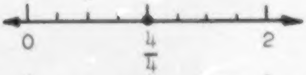
f. 

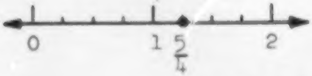
g. 

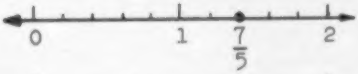
h. 

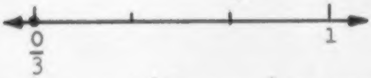
2. a. 

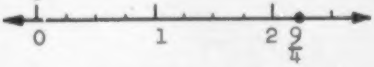
b. 

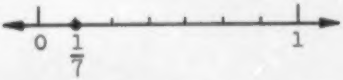
c. 

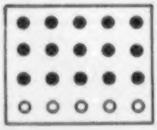
d. 

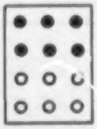
e. 

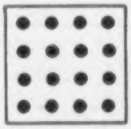
f. 

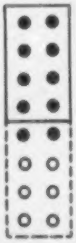
g. 

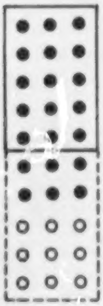
h. 

3. a. 

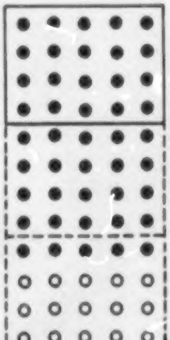
b. 

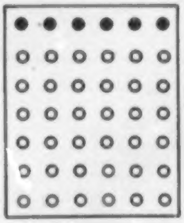
c. 

d. 

e. 

f. (empty)

g. 

h. 

4. a. $\frac{6}{4}$

b. $\frac{2}{8}$

c. $\frac{3}{8}$

d. $\frac{4}{6}$

e. $\frac{6}{10}$

f. $\frac{3}{2}$

g. not an appropriate model

h. not an appropriate model

i. not an appropriate model

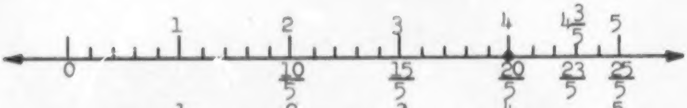
j. $\frac{6}{6}$

5. a. A , $\frac{1}{2}$ or $\frac{2}{4}$; B , $\frac{1}{4}$; C , $\frac{3}{4}$; D , $\frac{1}{3}$; E , $\frac{2}{3}$

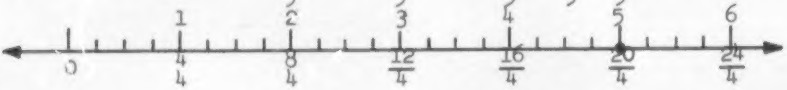
b. less than, since B lies to the left of D while 1 lies to the right of 0 .

c. $\frac{1}{2}$ or $\frac{2}{4}$

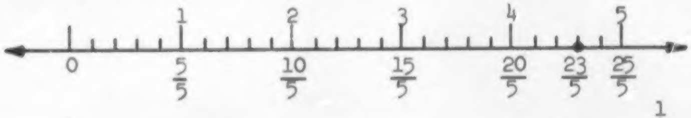
6. a.


1
2
3
4
4 3/5
5
0
10/5
15/5
20/5
23/5
25/5

b.


1
2
3
4
5
6
0
4/4
8/4
12/4
16/4
20/4
24/4

c.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5
0
5/5
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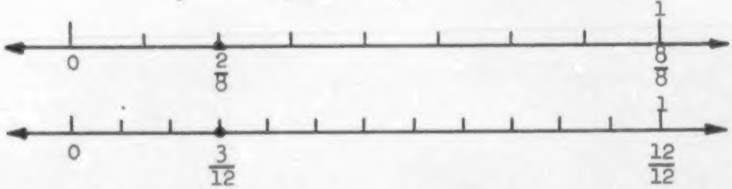
20/5
23/5
25/5

7.


1
0
2/8
8/8
1
0
3/12
12/12

8. $\frac{11}{4}$, $\frac{7}{12}$, $\frac{12}{13}$, $\frac{7}{412}$, $\frac{412}{7}$, $\frac{2}{3}$

9. a. $\frac{1}{25} < \frac{1}{24}$

b. $\frac{11}{24} < \frac{12}{26}$

c. $\frac{7}{8} > \frac{5}{6}$

d. $\frac{17}{32} > \frac{1}{2}$

e. $\frac{13}{26} = \frac{9}{18}$

10. a. $1\frac{3}{4}$ b. $1\frac{7}{8}$ c. $2\frac{3}{9} = 2\frac{1}{3}$ d. $2\frac{4}{15}$

e. $4\frac{8}{12} = 4\frac{2}{3}$

Chapter 12

1. d. and e. only

2. a. one pound ; b. 18 ; c. 18 pounds

3. a. 9 chalk pieces; b. 9 ; c. one chalk piece

4. c. and e. only

5. a. 1 , 1 , 1
   b. 45 ; 4
   c. 4 is not the sum of the measures of the sides in feet.
   d. The measure of a perimeter of a polygon is obtained by the measure of a segment which is the union of non-overlapping segments congruent to the sides of the polygon. Each side of the triangle is longer than one foot, and therefore the errors account for the extra foot in the perimeter.

6. The measure of $\overline{CD}$ is 1 . The measure of $\overline{EF}$ is 1 . No. No. Congruent segments must have the same measure, regardless of the unit. However, segments may have the same measure without being congruent. It is necessary, however, that with reference to some unit, non-congruent segments must have different measures. In the case of $\overline{CD}$ and $\overline{EF}$, the measure of $\overline{CD}$ is 6 and the measure of $\overline{EF}$ is 8 if the unit is G — H

Chapter 13

1. These answers are illustrative; others are possible.

a. $\begin{array}{l} (30 + 4) \\ \underline{\times \qquad 7} \\ 210 + 28 \\ 230 + 8 = 238 \end{array}$ or $\begin{array}{r} 34 \\ \underline{\times 7} \\ 28 \\ \underline{210} \\ 238 \end{array}$ b. $\begin{array}{r} 48 \\ \underline{\times 6} \\ 48 \\ \underline{240} \\ 288 \end{array}$ or $\begin{array}{r} 48 \\ \underline{\times 6} \\ 288 \end{array}$

c. $\begin{array}{l} (20 + 8) \\ \underline{\times \qquad 9} \\ 180 + 72 \\ 250 + 2 = 252 \end{array}$ or $\begin{array}{r} 20 \\ \underline{\times 9} \\ 180 \end{array}$ $\begin{array}{r} 8 \\ \underline{\times 9} \\ 72 \end{array}$ $\begin{array}{r} 180 \\ \underline{+ 72} \\ 252 \end{array}$

d. $\begin{array}{r} 54 \\ \underline{\times 8} \\ 32 \\ \underline{40} \\ 432 \end{array}$ or $\begin{array}{r} 54 \\ \underline{\times 8} \\ 432 \end{array}$

2. a. 3[illegible] $= (6 \times 6) + 2$; also, $38 \div 6 = 6\frac{2}{6} = 6\frac{1}{3}$

b. $99 = (24 \times 4) + 3$; also, $99 \div 4 = 24\frac{3}{4}$

c. $125 = (15 \times 8) + 5$; also, $125 \div 8 = 15\frac{5}{8}$

d. $84 = (28 \times 3)$; also, $84 \div 3 = 28$

3. a. $\begin{array}{r} 48 \\ 7\overline{)342} \\ \underline{28} \\ 62 \\ \underline{56} \\ 6 \end{array}$ b. $\begin{array}{r} 48 \\ 7\overline{)342} \\ \underline{28} \\ 62 \\ \underline{56} \\ 6 \end{array} \begin{array}{c} \\ \\ \longrightarrow \\ \\ \longrightarrow \\ \longrightarrow \end{array} \begin{array}{r} \\ 7\overline{)342} \\ \underline{-\ 280} \\ 62 \\ \underline{-\ 56} \\ 6 \end{array} \begin{array}{r} \\ \\ 40 \\ \\ \underline{8} \\ 48 \end{array}$

$\therefore\ 342 \div 7 = 48\frac{6}{7}$ $\qquad \therefore\ 342 = (48 \times 7) + 6$

4. $n \times b \leq a$ in order to assure that the multiple of b is less than or equal to a .

If $n \times b > a$, the subtraction would not be meaningful.

$r < b$ in order to be sure that n is as large as it can be.

If $r = b$, the quotient would be one more than n ;

if $r > b$, the quotient would be at least one more than n with or without a remainder.

GLOSSARY

Mathematical terms and expressions are frequently used with different meanings and connotations in the different fields or levels of mathematics. The following glossary explains some of the mathematical words and phrases as they are used in this book and in the K-3 texts. These are not intended to be formal definitions. More explanations, as well as figures and examples, may be found in the book.

A

ADDEND. If 8 is the sum of 2 and 6, then 2 and 6 are each an addend of 8.

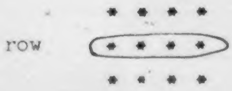
ADDITION. An operation on two numbers to obtain a third number called their sum.

ALGORITHM. A numerical expression of a computation using properties of addition and multiplication and characteristics of a system of numeration to determine the standard name for a sum, difference, product, or quotient.

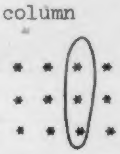
ANGLE. The union of two rays which have the same endpoint but which do not lie in the same line.

AS MANY AS; AS MANY MEMBERS AS. If two sets are equivalent, then one set is said to have as many members as the other set.

ARRAY. An orderly arrangement of rows and columns which may be used as a physical model to interpret multiplication of whole numbers. For example,



3 × 4



3 × 4

A rectangular array is implied by ARRAY unless otherwise specified.

ASSOCIATIVE PROPERTY OF ADDITION. When three numbers are added in a given order, the sum is independent of the grouping. That is, for any three numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

ASSOCIATIVE PROPERTY OF MULTIPLICATION. When three numbers are multiplied in a given order, the product is independent of the grouping. That is, for any three numbers a, b, and c,

$$(a \times b) \times c = a \times (b \times c).$$

B

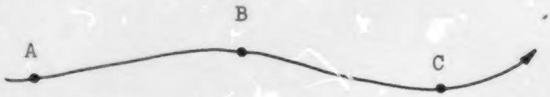
BASE (of a geometric figure). A particular side or face of a geometric figure. For example, the base of a parallelogram is one of the sides; the base of a square pyramid is the face that is the square region.

BASE (of a numeration system). A basic number in terms of which we affect groupings within the system. Ten is the base of a decimal system and two is the base of a binary system.

BASIC FACTS (addition, multiplication, subtraction, division). Basic addition and multiplication facts are sentences which express two names for the sums and products of all ordered pairs of whole numbers less than 10. One name expresses the sum or product, using the ordered pair. The other name expresses the sum or product, using the standard name. For example, $2 + 4 = 6$ is a basic addition fact; $3 \times 4 = 12$ is a basic multiplication fact.

Basic subtraction and division facts express the differences and quotients for any ordered pairs of whole numbers a and b, such that $a - b = c$ if $c + b = a$ and $a \div b = c$, such that $c \times b = a$, where b and c are both whole numbers less than 10.

BETWEEN. If a curve passes through three points A, B, and C,



then B is between A and C. When a curve is not specified, it is understood that the curve is a line or a segment through the points.

If for three numbers a, b, and c, $a < b$ and $b < c$, then b is between a and c.

C

CARTESIAN PRODUCT. If, for two given sets, $A = \{a, b, c\}$ and $B = \{1, 2\}$, then the Cartesian product (product set) of A and B is expressed as

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

CIRCLE. The set of all points in a plane which are the same distance from a given point. Alternatively, a circle is a simple closed curve having a point O in its interior such that, if A and B are any two points of the circle, $\overline{OA}$ is congruent to $\overline{OB}$.

CLOSED CURVE. A curve whose starting and endpoints are the same.

COLUMN. See ARRAY.

COMMUTATIVE PROPERTY OF ADDITION. When two numbers are added, their sum is independent of the order of the addends. For any two numbers a and b, $a + b = b + a$.

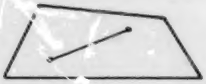
COMMUTATIVE PROPERTY OF MULTIPLICATION. When two numbers are multiplied, their product is independent of the order of the factors. For any two numbers a and b, $a \times b = b \times a$.

COMPLEMENT OF A SET. See REMAINING SET.

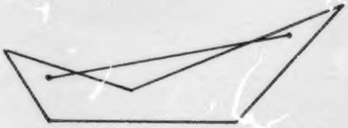
CONGRUENCE. The relationship between two geometric figures which have exactly the same size and shape.

208

CONVEX POLYGON. A polygon is said to be convex if the segment determined by any two interior points lies entirely in the interior.



The polygon below is not convex. It is said to be concave.



CONVEX SET. A set is said to be convex if a segment determined by any two points of the set lies entirely in the set.

COORDINATE. The name of a point on the number line.

COUNTING. The pairing of objects in a set with the numerals in the equivalent standard set.

COUNTING NUMBERS. Members of {1, 2, 3, 4, ...}; that is, the whole numbers with the exception of 0.

CURVE. A curve is a set of points followed in going from one point to another.

D

DENOMINATOR. The second member of the ordered pair of whole numbers associated with a fraction. It is the number (nonzero) of congruent parts or equivalent subsets into which a unit has been divided.

DIFFERENCE. The number which is assigned to an ordered pair of numbers under subtraction. 4 is the difference of 6 and 2.

DIGIT. Any one of the numerals in the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.

DISJOINT SETS. Two or more sets which have no members in common.

DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION. A joint property of multiplication and addition. For any three numbers $\underline{a}$, b, and c, then

$$a \times (b + c) = (a \times b) + (a \times c).$$

DIVISION. An operation on two numbers, $\underline{a}$ and b, such that $a \div b = n$ if and only if $n \times b = a$.

E

EDGE. The intersection of two polygonal regions which are faces of the surface of a solid. Where two faces meet is an edge of the solid.

For cylinders and cones, the boundary of a face is an edge.

EMPTY SET. The set which has no members.

EQUAL. $A = B$ means that A and B are names for the same thing. For example, $5 - 2 = 3$ expresses two names for the difference of 5 and 2; also, $A = B$ if A and B are sets consisting of the same members.

EQUATION. A sentence which expresses an equality. Open number sentences are called equations if the verb is "equals", or "is equal to".

EQUIVALENT. Two or more sets are said to be equivalent if their members can be put into a one-to-one correspondence; that is, each element of A is paired with exactly one element of B and no element of B is left unpaired.

EVEN NUMBER. An integer which can be expressed as $2 \times n$ where n is an integer.

EXPANDED FORM. The numeral 532 written as

$$(5 \times 10 \times 10) + (3 \times 10) + (2 \times 1)$$

or as $500 + 30 + 2$

is said to be written in expanded form.

EXTERIOR (OUTSIDE) OF A SIMPLE CLOSED PLANE CURVE. The subset of the plane which excludes both the simple closed curve and the subset of the plane enclosed by the plane geometric figure.

EXTERIOR (OUTSIDE) OF A SIMPLE CLOSED SURFACE. The subset of points in space which excludes both the simple closed surface and the subset of points enclosed by the surface.

F

FACTOR. If 10 is the product of 2 and 5, then 2 and 5 are both factors of 10.

FEWER THAN; FEWER (MEMBERS) THAN. If, in pairing the elements of A with those of B, there is an element of B which is not paired with any element of A, then A has fewer members than B.

FINITE SET. A set is finite if there is a whole number that will answer the question, "How many elements are there in the set?"

The notation $\{0, 1, 2, 3, 4, 5, 6\}$ describes the set of the first seven whole numbers, a finite set.

FRACTION. The numeral of the form $\frac{a}{b}$ where b is not equal to 0.

G

GREATER THAN. Associated with the relation "has more members than" for sets is the relation "is greater than" for numbers. For example, "$9 > 8$" is read "9 is greater than 8". For any two numbers a and b, $a > b$, if $a - b$ is a positive number.

H

HEXAGON. A polygon with six sides.

I

IDENTITY ELEMENT. The number 0 is the identity element for addition because the sum of 0 and any given number is the given number; that is, $0 + a = a$.

The number 1 is the identity element for multiplication because the product of 1 and any given number is the given number; that is, $1 \times a = a$.

IDENTITY PROPERTY. The property which states that there is an identity element under a particular operation.

INFINITE SET. A set is infinite if there is no whole number that will answer the question, "How many elements are there?"

The notation $\{0, 1, 2, 3, 4, 5, 6, \ldots\}$ describes the set of whole numbers, an infinite set.

INTERIOR (INSIDE) OF A SIMPLE CLOSED PLANE CURVE. The subset of the plane enclosed by the simple closed curve.

INTERIOR OF A SIMPLE CLOSED SURFACE. The subset of points enclosed by the simple closed surface.

INTERSECTION. The set of points common to two or more sets of points.

INVERSE (DOING AND UNDOING) OPERATIONS. Two operations such that one "undoes" what the other one "does". For example, putting on a jacket and taking it off are inverse operations.

J

JOIN; UNION. The union of two disjoint sets to form a third set, whose members are all the elements in each of the two sets. For example,

if $A = \{red, blue, green\}$, and $B = \{white, orange\}$,
then $A \cup B = \{red, blue, green, white, orange\}$.

L

LENGTH. The common property of congruent segments. We approximate length by measurement or comparison with specified unit segments, in the length approximated by the measurement 5 miles, 5 is the measure and the unit is the mile.

LESS THAN. Associated with the relation "has fewer members than" for sets, is the relation "is less than" for numbers. For example, "$2 < 5$" is read "2 is less than 5". For any two numbers a and b, $a < b$ if $b - a$ is a positive number.

LINE. A line is conceived of as the unlimited extension of a given segment in both directions.

LINE SEGMENT. A special case of the curves between two points. It may be represented by a string stretched tautly between its two endpoints.

LINEAR SCALE. A scale is a number line with the segment from 0 to 1 congruent to the unit being used.

M

MATCH. Two sets match if their members can be put in one-to-one correspondence.

MEASURE. A number assigned to a geometric figure indicating its size (length, area, volume, time, etc.) with respect to a specific unit. For example, the measure in inches of $\overline{AB}$ is 3.

MEMBER (of a set). An object in a set.

MISSING ADDEND. If 8 is the sum of 2 and n, then n is the missing addend.

MISSING FACTOR. If 10 is the product of 2 and n, then n is the missing factor.

MORE (MEMBERS) THAN. If, in pairing the elements of A with those of B, there is at least one member of B which is not paired with any element of A, then B has more members than A.

MULTIPLICATION. An operation on two numbers to obtain a third number called their product.

N

NATURAL NUMBERS. See COUNTING NUMBERS.

NEGATIVE NUMBER. Any number that is less than 0.

NUMBER LINE. A line marked off at intervals congruent to a chosen unit segment such that: there is a starting point associated with the number 0; the endpoint of successive intervals are labeled according to the counting numbers in their natural order.

NUMBER (PROPERTY) OF A SET. The number of elements in the set. The number property of A is written N(A), where A is a set.

NUMERAL. A name for a number.

NUMERATION SYSTEM. A system for naming numbers. The Roman numeral system and the decimal system are systems of numeration.

NUMERATOR. The first number of the ordered pair of whole numbers associated with a fraction. It is the number of congruent parts or equivalent subsets being considered.

O

ODD NUMBERS. An integer which cannot be expressed as $2 \times n$, where n is an integer.

ONE-TO-ONE CORRESPONDENCE. A pairing between two sets A and B, which associates with each element of A a single element of B, and with each element of B a single element of A.

OPERATION. The association of a third number with an ordered pair of numbers is a binary operation. For example, in the operation of addition, the number 7 is associated with the pair of numbers 5 and 2.

In general, an operation is the association of a unique element to each element of a given set, or to each combination of elements, one from each of the given sets.

ORDER. A property of a set of numbers which permits one to say whether $\underline{a}$ is less than b, greater than b, or equal to b, where $\underline{a}$ and b are members of the set.

P

PAIRING. A correspondence between an element of one set and an element of another set.

PARTITION. See PARTITIONING.

PARTITIONING. Partitioning a finite set means separating the set into disjoint subsets so that the union of the subsets is the given set.

In partitioning an infinite set such as a line segment, the subsets need not be disjoint. However, any two subsets have at most the points of separation in common.

The separation is the partition.

PENTAGON. A polygon with five sides.

PLACE VALUE. A value given to a certain position in a numeral. Thus, the place assigns to the digit 2 in 235 the value 200.

PLANE. A particular set of points which can be thought of as the extension of a flat surface, such as the surface of a table.

PLANE REGION. The union of a simple closed plane curve and its interior.

POLYGON. A simple closed curve which is the union of three or more line segments.

PRODUCT. The third number associated with an ordered pair of numbers by multiplication. For example, 8 is the product of 2 and 4.

PRODUCT SET. See CARTESIAN PRODUCT.

Q

QUADRILATERAL. A polygon with four sides.

QUOTIENT. The third number associated with an ordered pair of numbers by division. For example, 12 is the quotient of 48 and 4.

R

RATIO. A relationship between an ordered pair of numbers *a* and b where $b \neq 0$. The ratio may be expressed by a : b or by $\frac{a}{b}$.

RATIONAL NUMBER. A number which may be expressed as $\frac{a}{b}$ or $-\frac{a}{b}$, where *a* and b are whole numbers with $b \neq 0$.

RAY. Ray AB is the union of segment AB and all points C such that B is between A and C.

RECTANGLE. A quadrilateral with four right angles.

REGION. See PLANE REGION AND SOLID REGION.

REMAINDER; REMAINDER SET. See REMAINING SET.

REMAINING SET; REMAINDER (SET). If B is a subset of A, all members of A which are not members of B are members of the remaining or remainder set. The complement of B relative to A is the remaining set.

RENAMING. Using another name for the same number. For example, 34 can be renamed as 30 + 4, 20 + 14, 2 × 17, and so on.

RIGHT ANGLE. One of two congruent angles determined by a line and a ray having a point in the line as endpoint.

ROUND. A shape which has no corners or sides.

ROW. See ARRAY.

RULER. A straightedge on which a scale using a standard unit has been marked.

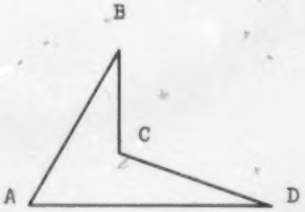
S

SCALE. See LINEAR SCALE.

SEGMENT. See LINE SEGMENT.

SENTENCE. A statement, such as "$9 + 5 = 14$" is a number sentence; it connects sets of numerical and operational symbols showing a relation between the sets of symbols. Examples of symbols relating the sets are: $=$, $<$, and $>$. These symbols act as verbs in the sentences.

SIDE. A segment of a polygon that is contained in no segment of the polygon other than itself. For example, $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, are sides of the quadrilateral illustrated at the right.



SIMPLE CLOSED CURVE. A closed curve which does not intersect itself.

SOLID. A geometric figure that is not a subset of any one plane.

SOLID REGION. The union of a simple closed surface and its interior.

SQUARE. A rectangle whose sides are congruent.

STANDARD SET. One of the sets of ordered numerals such as $\{1, 2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$.

STANDARD UNIT. A standard unit is a unit of measure "officially" agreed upon or accepted as a standard. Examples are: inch, meter, gram.

SUBSET. Given two sets A and B, B is a subset of A if every member of B is also a member of A.

SUBTRACTION. An operation on two numbers a and b to obtain a third number n, called the difference such that $a - b = n$ if $n + b = a$.

SUM. The third number associated with an ordered pair of numbers by addition. For example, 6 is the sum of 2 and 4.

T

TIMES. The word associated with × to indicate the operation, multiplication.

TRIANGLE. A polygon with three sides.

U

UNION. The operation that associates with two sets, a third set consisting of all the members in each of two sets. For example,

if A = {red, blue, green, white, yellow} and
B = {blue, white, orange},
then $A \cup B$ = {red, blue, green, white, yellow, orange}.

UNIT. A prototype from which the measure is obtained by comparison. For example, the unit in measuring length is a segment; the unit for area is a square region.

V

VERTEX OF AN ANGLE. The common endpoint of its two rays.

VERTEX OF A POLYGON. If two sides have a point in common then this common point is a vertex. The plural of vertex is vertices.

VERTEX OF A PRISM OR PYRAMID. If three or more edges have a point in common, then the common point is a vertex.

W

WHOLE NUMBER. The property common to a set of equivalent sets. Members of {0, 1, 2, 3, ...}.